STAT611 **Exam 2** **November 8, 2011**

1. Suppose $A_{p\times p}$ is a symmetric, idempotent, and positive definite matrix. Provide as much additional information about A as possible.

2. Suppose $\underline{y}$ is an $n \times 1$ random vector of responses, and X is an $n \times p$ known design matrix.

   (a) State the Aitken Model.

   (b) Provide an expression for the BLUE of an estimable function $\underline{c}'\underline{\beta}$. (No derivations are necessary here. Just write down the answer.)

   (c) Complete the following statement. (Again, no derivations are necessary.)

   The ordinary least squares estimator of any estimable function $\underline{c}'\underline{\beta}$ is the BLUE of $\underline{c}'\underline{\beta}$ if and only if ____.

3. Suppose $\underline{y} = X\underline{\beta} + \underline{\varepsilon}$, where $\mathrm{E}(\underline{\varepsilon}) = \underline{0}$.

   (a) Prove that $\underline{c}'\underline{\beta}$ is an estimable function if and only if the equations $X'X\underline{r} = \underline{c}$ are consistent. These equations are known as the *conjugate normal equations*.

   (b) Prove that $\underline{r}'X'\underline{y}$ is the least squares estimator of an estimable function $\underline{c}'\underline{\beta}$ if and only if $\underline{r}$ is a solution to the conjugate normal equations.

4. Suppose that $\underline{y} \sim N(W\underline{\alpha}, \sigma^2 I)$, where $\underline{\alpha}$ is a vector of unknown real parameters, σ^2 is an unknown positive variance, and W is an unknown fixed design matrix. Instead of using the unknown W as the design matrix, suppose we use a known matrix X as our design matrix. Furthermore, suppose we proceed with our analysis of the data by assuming (perhaps incorrectly) that $\underline{y} \sim N(X\underline{\beta}, \sigma^2 I)$, where $\underline{\beta}$ is a vector of unknown real parameters and σ^2 is an unknown positive variance.

   (a) Complete the following with a simply stated condition involving matrix column spaces, and prove that the resulting statement is true.

   > The least squares estimator of $\mathrm{E}(\underline{y})$ computed using design matrix X is unbiased for $\mathrm{E}(\underline{y})$ if and only if ____.

   (b) Prove that the least squares estimator of $\underline{a}'\mathrm{E}(\underline{y})$ computed using design matrix X is unbiased for $\underline{a}'\mathrm{E}(\underline{y})$ if and only if $\underline{a} \in \mathcal{N}(P_W(I - P_X))$.

   (c) Show by example that the least squares estimator of $\underline{a}'\mathrm{E}(\underline{y})$ computed using design matrix X may be unbiased for $\underline{a}'\mathrm{E}(\underline{y})$ even though the least squares estimator of $\mathrm{E}(\underline{y})$ computed using design matrix X is a biased estimator of $\mathrm{E}(\underline{y})$.

   (d) Derive the distribution of $\hat{\sigma}^2 = \underline{y}'(I - P_X)\underline{y}/(n - \mathrm{rank}(X))$.

5. Suppose the Gauss-Markov Model holds for $\underline{y} = X\underline{\beta} + \underline{\varepsilon}$. Let

$$X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \text{ and suppose } \underline{y} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}.$$

   Prove that the BLUE of $\underline{\beta}$ subject to the constraint $\beta_1 + 2\beta_2 = 0$ takes the value $[2, -1]'$ in this case.