STAT611 **Exam 1** **September 25, 2007**

1. A matrix $A : n \times n$ is said to be an orthogonal projector if A is both idempotent and symmetric. Let $\underline{y}$ denote an arbitrary element of $\mathbb{R}^n$. Prove that if A is symmetric and idempotent, then $A\underline{y}$ is the closest point in the range of A to the point $\underline{y}$. (Note that you may use any results that have already been established in our course notes to prove this fact, so your proof here may be quite short.)

2. Suppose A is an $m \times n$ matrix. Recall that the null space of A is the set

$$\mathcal{N}(A) = \{\underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{0}\}.$$

Your textbook states a result about the dimension of $\mathcal{N}(A)$ relative to the dimension of $\mathcal{R}(A)$ which you do not need to know to solve the following problems. Furthermore, solutions to the following problems can be used to establish a proof of the fact. Thus, even if you know the fact, please solve the following problems without using the fact.

   (a) Show that $\mathcal{N}(A)$ is a vector space.

   (b) Find a matrix B such that $\mathcal{R}(B) = \mathcal{N}(A)$.

   (c) State the rank of B, and prove that your answer is correct.

   (d) Determine the dimension of $\mathcal{N}(A)$ and briefly explain your reasoning.

3. Consider the set $\mathcal{S} = \{\underline{x} \in \mathbb{R}^3 : 2x_1 + x_2 = 0\}$. Determine an expression for the projection of $\underline{y}$ onto $\mathcal{S}$ in terms of y_1, y_2, and y_3 (the components of $\underline{y}$).

4. Suppose λ_1 and λ_2 are non-zero eigenvalues of a symmetric matrix A. Furthermore, suppose $\lambda_1 \neq \lambda_2$. Use our definition of eigenvectors provided in class notes to show that if $\underline{x}_1$ and $\underline{x}_2$ are eigenvectors corresponding to λ_1 and λ_2, respectively, then $\underline{x}_1$ and $\underline{x}_2$ are orthogonal.

5. Suppose $\mathrm{rank(A)} = \mathrm{r_A}$, $\mathrm{rank(B)} = \mathrm{r_B}$, and $\mathrm{rank(C)} = \mathrm{r_C}$. Find the rank of the following matrix in terms of r_A, r_B, and r_C; and prove that your answer is correct.

$$\begin{bmatrix} B & BC \\ AB & ABC \end{bmatrix}$$

6. Consider the regression model $\underline{y} = X\underline{\beta} + \underline{e}$ that we have discussed in class. Suppose that $X = [\underline{x}_1, \underline{x}_2]$ and that $\mathrm{rank(X)} = 2$. Consider the following claim:

   Let $\hat{\underline{\beta}}$ denote a solution to the normal equations. The first component of $\hat{\underline{\beta}}$ can be found by

   (a) regressing $\underline{x}_1$ on $\underline{x}_2$,

   (b) obtaining $\underline{z} =$ residuals from the regression in step (a), and

   (c) finding a solution to the normal equations corresponding to the regression of $\underline{y}$ on $\underline{z}$.

   Is this claim true? Provide a proof to support your answer.