Confidence Intervals and Confidence Regions for Estimable Functions

Suppose

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$

where

$$\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2\boldsymbol{I}).$$

Suppose $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable.

Prove that

$$t \equiv \frac{\boldsymbol{c}'\hat{\boldsymbol{\beta}} - \boldsymbol{c}'\boldsymbol{\beta}}{\sqrt{\hat{\sigma}^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{c}}}$$

has a t-distribution with $n - r$ degree of freedom.

From previous results, we have

$$\boldsymbol{c}'\hat{\boldsymbol{\beta}} \sim N(\boldsymbol{c}'\boldsymbol{\beta}, \sigma^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{c}).$$

Thus,

$$\frac{\boldsymbol{c}'\hat{\boldsymbol{\beta}} - \boldsymbol{c}'\boldsymbol{\beta}}{\sqrt{\sigma^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{c}}} \sim N(0, 1).$$

Also from previous results,

$$\frac{\hat{\sigma}^2}{\sigma^2} = \boldsymbol{y}' \left(\frac{\boldsymbol{I} - \boldsymbol{P_X}}{\sigma^2(n-r)} \right) \boldsymbol{y} \sim \frac{\chi^2_{n-r}}{n-r}.$$

Note that

$$\begin{aligned} \boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'(\boldsymbol{I} - \boldsymbol{P_X}) &= \boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^-(\boldsymbol{X}' - \boldsymbol{X}'\boldsymbol{P_X}) \\ &= \boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^-(\boldsymbol{X}' - \boldsymbol{X}') \\ &= \boldsymbol{0}'. \end{aligned}$$

Thus

$$\boldsymbol{c}'\hat{\boldsymbol{\beta}} = \boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y}$$

and

$$\hat{\sigma}^2 = \boldsymbol{y}'\left(\frac{\boldsymbol{I} - \boldsymbol{P_X}}{n - r}\right)\boldsymbol{y}$$

are independent by Result 5.16.

It follows that

$$\frac{\boldsymbol{c}'\hat{\boldsymbol{\beta}} - \boldsymbol{c}'\boldsymbol{\beta}}{\sqrt{\hat{\sigma}^2 \boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^- \boldsymbol{c}}} = \frac{\boldsymbol{c}'\hat{\boldsymbol{\beta}} - \boldsymbol{c}'\boldsymbol{\beta}}{\sqrt{\sigma^2 \boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^- \boldsymbol{c}}} \Bigg/ \sqrt{\hat{\sigma}^2/\sigma^2}$$

$$\sim \quad t_{n-r}.$$

Let $t_{n-r,\alpha/2}$ denote the upper $\alpha/2$ quantile of a t-distribution with $n-r$ DF.

It follows that

$$1-\alpha \;=\; \mathbb{P}\left(-t_{n-r,\alpha/2} \leq \frac{\boldsymbol{c}'\hat{\boldsymbol{\beta}}-\boldsymbol{c}'\boldsymbol{\beta}}{\sqrt{\hat{\sigma}^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{c}}} \leq t_{n-r,\alpha/2}\right)$$

$$= \mathbb{P}\left(\boldsymbol{c}'\hat{\boldsymbol{\beta}} - t_{n-r,\alpha/2}\sqrt{\hat{\sigma}^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{c}} \leq \boldsymbol{c}'\boldsymbol{\beta} \leq \boldsymbol{c}'\hat{\boldsymbol{\beta}} + t_{n-r,\alpha/2}\sqrt{\hat{\sigma}^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{c}}\right).$$

Thus, a $100(1-\alpha)\%$ confidence interval for $\boldsymbol{c}'\boldsymbol{\beta}$ is

$$\left(\boldsymbol{c}'\hat{\boldsymbol{\beta}} - t_{n-r,\alpha/2}\sqrt{\hat{\sigma}^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{c}}, \boldsymbol{c}'\hat{\boldsymbol{\beta}} + t_{n-r,\alpha/2}\sqrt{\hat{\sigma}^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{c}}\right).$$

From previous results, we know that

$$\frac{(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{C}\boldsymbol{\beta})'(\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}')^{-1}(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{C}\boldsymbol{\beta})}{q\hat{\sigma}^2} \sim F_{q,n-r}$$

for estimable $\boldsymbol{C}\boldsymbol{\beta}$ with $\underset{q\times p}{\boldsymbol{C}}$ of rank q.

Let $F_{q,n-r,\alpha}$ denote the upper α quantile of an F distribution with q and $n-r$ DF.

It follows that

$$\mathbb{P}\left[\frac{(\boldsymbol{C}\hat{\boldsymbol{\beta}}-\boldsymbol{C}\boldsymbol{\beta})'(\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}')^{-1}(\boldsymbol{C}\hat{\boldsymbol{\beta}}-\boldsymbol{C}\boldsymbol{\beta})}{q\hat{\sigma}^2} \leq F_{q,n-r,\alpha}\right] = 1-\alpha.$$

$\therefore$ the set

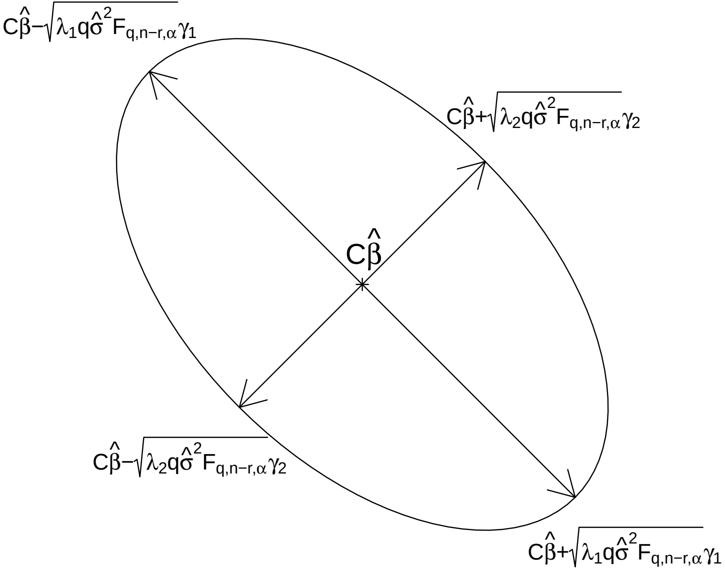
$$\{\boldsymbol{\theta} \in \mathbb{R}^q : (\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{\theta})'(\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}')^{-1}(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{\theta}) \leq q\hat{\sigma}^2 F_{q,n-r,\alpha}\}$$

is a $100(1-\alpha)\%$ confidence region for $\boldsymbol{C\beta}$.

This confidence region is an ellipsoid centered at $\boldsymbol{C}\hat{\boldsymbol{\beta}}$ with axes

$$\pm\sqrt{\lambda_j q \hat{\sigma}^2 F_{q,n-r,\alpha}}\gamma_j,$$

where $\lambda_1, \ldots, \lambda_q$ are the eigenvalues of $\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{C}'$ and $\gamma_1, \ldots, \gamma_q$ are the corresponding eigenvectors.


$C\hat{\beta} - \sqrt{\lambda_1 q \hat{\sigma}^2 F_{q,n-r,\alpha}}\gamma_1$
$C\hat{\beta} + \sqrt{\lambda_2 q \hat{\sigma}^2 F_{q,n-r,\alpha}}\gamma_2$
$C\hat{\beta}$
$C\hat{\beta} - \sqrt{\lambda_2 q \hat{\sigma}^2 F_{q,n-r,\alpha}}\gamma_2$
$C\hat{\beta} + \sqrt{\lambda_1 q \hat{\sigma}^2 F_{q,n-r,\alpha}}\gamma_1$

Suppose

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$$

for $i = 1, 2, 3;\ j = 1, 2$, where

$$\varepsilon_{11}, \ldots, \varepsilon_{32} \overset{i.i.d.}{\sim} N(0, \sigma^2).$$

Suppose

$$\boldsymbol{y} = \begin{bmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \\ y_{31} \\ y_{32} \end{bmatrix} = \begin{bmatrix} 12 \\ 14 \\ 5 \\ 9 \\ 5 \\ 7 \end{bmatrix}.$$

Find a 95% confidence region for

$$\begin{bmatrix} \alpha_1 - \alpha_2 \\ \alpha_2 - \alpha_3 \end{bmatrix}.$$

$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{\beta} = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\boldsymbol{X}'\boldsymbol{X} = \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}, \qquad (\boldsymbol{X}'\boldsymbol{X})^- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$\boldsymbol{X}'\boldsymbol{y} = \begin{bmatrix} 52 \\ 26 \\ 14 \\ 12 \end{bmatrix}, \quad \hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y} = \begin{bmatrix} 0 \\ 13 \\ 7 \\ 6 \end{bmatrix}$$

$$\hat{\boldsymbol{y}} = \boldsymbol{X}\hat{\boldsymbol{\beta}} = \begin{bmatrix} 13 \\ 13 \\ 7 \\ 7 \\ 6 \\ 6 \end{bmatrix}, \quad \hat{\boldsymbol{\varepsilon}} = \boldsymbol{y} - \hat{\boldsymbol{y}} = \begin{bmatrix} -1 \\ 1 \\ -2 \\ 2 \\ -1 \\ 1 \end{bmatrix}.$$

$$
\begin{aligned}
\hat{\sigma}^2 &= \frac{\hat{\varepsilon}'\hat{\varepsilon}}{n-r} \\
&= \frac{(-1)^2 + 1^2 + (-2)^2 + 2^2 + (-1)^2 + 1^2}{6-3} \\
&= 4.
\end{aligned}
$$

$$
C = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad C\hat{\beta} = \begin{bmatrix} 13 & -7 \\ 7 & -6 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}
$$

$$
C(X'X)^{-}C' = \begin{bmatrix} 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}.
$$

$$
\begin{aligned}
|\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}' - \lambda\boldsymbol{I}| &= \left| \begin{bmatrix} 1-\lambda & -1/2 \\ -1/2 & 1-\lambda \end{bmatrix} \right| \\
&= (1-\lambda)^2 - 1/4.
\end{aligned}
$$

$$
\begin{aligned}
(1-\lambda)^2 - 1/4 = 0 &\iff |1-\lambda| = 1/2 \\
&\iff \lambda = 1.5 \text{ or } \lambda = 0.5.
\end{aligned}
$$

Eigenvalues of $\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}'$ are $\lambda_1 = 1.5, \lambda_2 = 0.5$.

$$\begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{12} \end{bmatrix} = 1.5 \begin{bmatrix} \gamma_{11} \\ \gamma_{12} \end{bmatrix}$$

$$\iff \quad \gamma_{11} - \gamma_{12}/2 = 1.5\gamma_{11}$$

$$-\gamma_{11}/2 + \gamma_{12} = 1.5\gamma_{12}$$

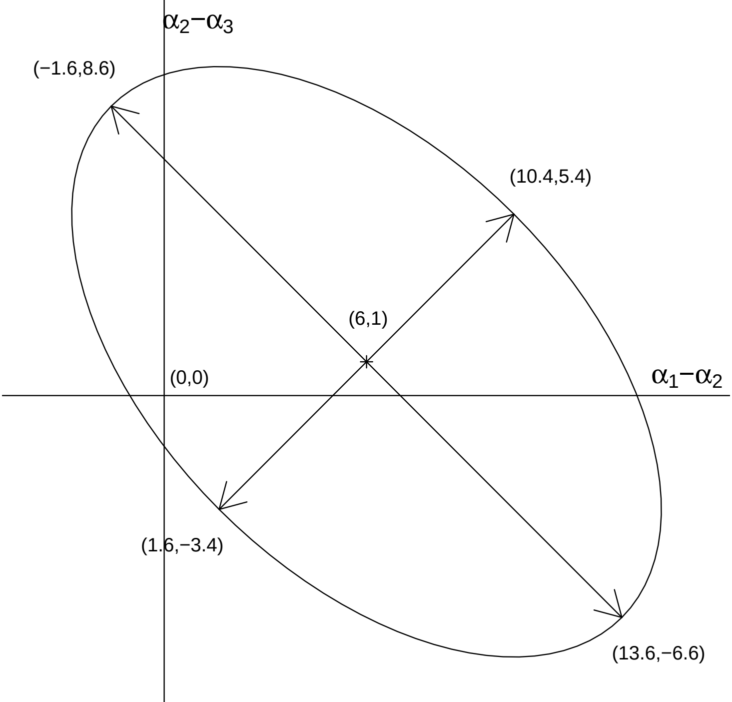
$$\implies \gamma_{11} = -\gamma_{12} \implies \boldsymbol{\gamma}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\implies \boldsymbol{\gamma}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

$$\boldsymbol{C}\hat{\boldsymbol{\beta}} \pm \sqrt{\lambda_i q \hat{\sigma}^2 F_{2,3,.05}}\boldsymbol{\gamma}_i$$

$$\begin{bmatrix} 6 \\ 1 \end{bmatrix} \pm \sqrt{(1.5)(2)(4)(9.55)} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \Longleftrightarrow \begin{bmatrix} -1.6 \\ 8.6 \end{bmatrix}, \begin{bmatrix} 13.6 \\ -6.6 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ 1 \end{bmatrix} \pm \sqrt{(0.5)(2)(4)(9.55)} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 1.6 \\ -3.4 \end{bmatrix}, \begin{bmatrix} 10.4 \\ 5.4 \end{bmatrix}.$$



$\alpha_2-\alpha_3$
(−1.6,8.6)
(10.4,5.4)
(6,1)
(0,0)
$\alpha_1-\alpha_2$
(1.6,−3.4)
(13.6,−6.6)