The F- and t-Distributions

Suppose U_1, U_2 are two independent random variables and

$$U_1 \sim \chi^2_{p_1}, U_2 \sim \chi^2_{p_2}.$$

Then

$$F = \frac{U_1/p_1}{U_2/p_2}$$

has the F-Distribution with p_1 and p_2 DF, denoted by

$$F \sim F_{p_1,\ p_2}.$$

Result 5.12:

The density of $F \sim F_{p_1,\, p_2}$ is

$$f_F(t) = \frac{\Gamma\left(\frac{p_1+p_2}{2}\right)\left(\frac{p_1}{p_2}\right)^{\frac{p_1}{2}}}{\Gamma\left(\frac{p_1}{2}\right)\Gamma\left(\frac{p_2}{2}\right)} t^{\frac{p_1}{2}-1}\left(1+\frac{p_1}{p_2}t\right)^{-\frac{p_1+p_2}{2}}.$$

Proof of Result 5.12:

HW problem. □

Suppose U_1 and U_2 are independent random variables and suppose

$$U_1 \sim \chi^2_{p_1}(\phi) \quad \text{and} \quad U_2 \sim \chi^2_{p_2}.$$

Then

$$F = \frac{U_1/p_1}{U_2/p_2}$$

has the noncentral F-distribution with p_1 and p_2 DF and NCP ϕ.

$(F \sim F_{p_1,\, p_2}(\phi))$

Result 5.13:

Suppose $W \sim F_{p_1,\, p_2}(\phi)$. Then for fixed p_1, p_2 and $c > 0$, $\mathbb{P}(W > c)$ is a strictly increasing function of ϕ.

Proof:

$$W \stackrel{d}{=} \frac{U_1/p_1}{U_2/p_2},$$

where U_1 independent of U_2, $U_1 \sim \chi^2_{p_1}(\phi)$, and $U_2 \sim \chi^2_{p_2}$.

Thus,

$$\begin{aligned}
\mathbb{P}(W > c) &= \mathbb{P}_\phi(W > c) \\
&= \mathbb{P}_\phi\left(\frac{U_1/p_1}{U_2/p_2} > c\right) \\
&= \mathbb{P}_\phi\left(U_1 > \frac{cp_1}{p_2}U_2\right) \\
&= \int_0^\infty \mathbb{P}_\phi\left(U_1 > \frac{cp_1}{p_2}u_2 \middle| U_2 = u_2\right) f_{U_2}(u_2)du_2
\end{aligned}$$

$$= \int_0^\infty g_\phi(u_2) f_{U_2}(u_2) du_2$$

where

$$g_\phi(u_2) = \mathbb{P}_\phi\left(U_1 > \frac{cp_1}{p_2}u_2 \middle| U_2 = u_2\right)$$
$$= \mathbb{P}_\phi\left(U_1 > \frac{cp_1}{p_2}u_2\right) \quad \text{by ind. of } U_1, U_2.$$

By Result 5.11,

$$g_{\phi_1}(u_2) < g_{\phi_2}(u_2) \quad \forall\, 0 \le \phi_1 < \phi_2 \quad \text{and} \quad \forall\, u_2 > 0.$$

Thus,

$$
\begin{aligned}
0 &< \int_0^\infty (g_{\phi_2}(u_2) - g_{\phi_1}(u_2)) f_{U_2}(u_2) du_2 \\
&= \int_0^\infty g_{\phi_2}(u_2) f_{U_2}(u_2) du_2 - \int_0^\infty g_{\phi_1}(u_2) f_{U_2}(u_2) du_2 \\
&= \mathbb{P}_{\phi_2}(W > c) - \mathbb{P}_{\phi_1}(W > c) \quad \forall\, 0 \le \phi_1 < \phi_2.
\end{aligned}
$$

$\therefore \mathbb{P}_\phi(W > c)$ is a strictly increasing function of ϕ. $\square$

Suppose

$$U \sim N(\mu, 1) \quad \text{and} \quad V \sim \chi^2_k.$$

If U and V are independent, then

$$T = \frac{U}{\sqrt{V/k}}$$

has the noncentral t-distribution with k DF and NCP μ. $(T \sim t_k(\mu))$

If $\mu = 0$, then $T = U/\sqrt{V/k}$ has (Student's) t-distribution with k degree of freedom and density

$$f_T(t) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\sqrt{\pi k}}\left(1 + \frac{t^2}{k}\right)^{-\frac{k+1}{2}}.$$

We use $T \sim t_k$ to indicate that T has Student's t-distribution with k DF.

Suppose $T \sim t_k(\mu)$.

Find the distribution of T^2.

If

$$T \sim t_k(\mu),$$

then

$$T \overset{d}{=} \frac{U}{\sqrt{V/k}},$$

where

$$U \sim N(\mu, 1) \text{ independent of } V \sim \chi^2_k.$$

Thus,

$$T^2 \overset{d}{=} \frac{U^2}{V/k}, \quad \text{with } U^2 \text{ independent of } V.$$

By Result 5.9,

$$U^2 \sim \chi_1^2(\mu^2/2).$$

Thus,

$$T^2 \stackrel{d}{=} \frac{U^2}{V/k} = \frac{U^2/1}{V/k}$$

has $F_{1,k}(\mu^2/2)$ distribution.