The Chi-Square Distribution

Suppose $\boldsymbol{Z} \sim N(\boldsymbol{0}, \underset{p\times p}{\boldsymbol{I}})$.

Then

$$U = \boldsymbol{Z}'\boldsymbol{Z} = \sum_{i=1}^{p} Z_i^2$$

has the Chi-Square Distribution with p Degrees of Freedom (DF).

This is denoted by $U \sim \chi^2_p$.

Find the MGF of $U \sim \chi^2_p$.

$$
\begin{aligned}
E(e^{tU}) &= E(e^{t\sum_{i=1}^{p} Z_i^2}) = E\left(\prod_{i=1}^{p} e^{tZ_i^2}\right) \\
&= \prod_{i=1}^{p} E(e^{tZ_i^2}) \\
&= \prod_{i=1}^{p} \int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{-1/2(z_i^2 - 2tz_i^2)} dz_i \\
&= \prod_{i=1}^{p} \int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{-1/2 z_i^2 (1-2t)} dz_i \\
&= \prod_{i=1}^{p} (1-2t)^{-1/2} \int_{-\infty}^{\infty} \left(\frac{2\pi}{1-2t}\right)^{-1/2} e^{-1/2 z_i^2 (1-2t)} dz_i \\
&= (1-2t)^{-p/2}.
\end{aligned}
$$

The density of $U \sim \chi^2_p$ is given by

$$f_U(u) = \frac{u^{(p-2)/2} e^{-u/2}}{\Gamma(p/2) 2^{p/2}},$$

where $\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy$ for $x > 0$.

Proof:

Homework problem. $\square$

Suppose

$$V \sim Poisson(\phi) \quad \text{and} \quad (U|V = j) \sim \chi^2_{p+2j}.$$

Then the unconditional distribution of U is the Noncentral Chi-Square Distribution with p DF and Noncentrality Parameter ϕ ($U \sim \chi^2_p(\phi)$).

If $U \sim \chi^2_p(\phi)$, the density of U is given by

$$f_U(u) = \sum_{j=0}^{\infty} \frac{u^{(p+2j-2)/2} e^{-u/2}}{\Gamma(\frac{p+2j}{2}) 2^{j+p/2}} \frac{\phi^j e^{-\phi}}{j!}.$$

The first factor in each term is the density of χ^2_{p+2j}, which is the conditional density of $(U|V=j)$.

The second factor in each term is the probability mass function of Poisson(ϕ), $(\mathbb{P}(V=j))$.

Result 5.5:

If $U \sim \chi^2_p(\phi)$, then the MGF of U is

$$M_U(t) = (1 - 2t)^{-p/2} e^{2\phi t/(1-2t)}.$$

Proof of Result 5.5:

$$
\begin{aligned}
E(e^{tU}) &= E(E(e^{tU}|V)) \\
&= E\left((1-2t)^{-(p+2V)/2}\right) \\
&= \sum_{j=0}^{\infty}(1-2t)^{-(p+2j)/2}\phi^j e^{-\phi}/j! \\
&= (1-2t)^{-p/2}\sum_{j=0}^{\infty}(1-2t)^{-j}\phi^j e^{-\phi}/j! \\
&= (1-2t)^{-p/2}\sum_{j=0}^{\infty}(\phi(1-2t)^{-1})^j e^{-\phi}/j!
\end{aligned}
$$

$$
\begin{aligned}
&= (1-2t)^{-p/2} e^{-\phi} \sum_{j=0}^{\infty} (\phi(1-2t)^{-1})^j / j! \\
&= (1-2t)^{-p/2} e^{-\phi} e^{\phi(1-2t)^{-1}} \\
&= (1-2t)^{-p/2} e^{\phi(1-2t)^{-1} - \phi} \\
&= (1-2t)^{-p/2} e^{\phi\left(\frac{1}{1-2t} - \frac{1-2t}{1-2t}\right)} \\
&= (1-2t)^{-p/2} e^{2\phi t/(1-2t)}.
\end{aligned}
$$

$\square$

Result 5.6:

If $U \sim \chi^2_p(\phi)$, then

$$E(U) = p + 2\phi \quad \text{and}$$
$$\text{Var}(U) = 2p + 8\phi.$$

Proof: HW problem. $\square$

Result 5.7:

If $U_1, \ldots, U_m$ are mutually independent and

$$U_i \sim \chi^2_{p_i}(\phi_i) \quad \forall\, i = 1, \ldots, m,$$

then

$$U = \sum_{i=1}^{m} U_i \sim \chi^2_p(\phi),$$

where

$$p = \sum_{i=1}^{m} p_i \quad \text{and} \quad \phi = \sum_{i=1}^{m} \phi_i.$$

Proof of Result 5.7:

$$
\begin{aligned}
M_U(t) &= E(e^{tU}) \\
&= E\left(e^{t\sum_{i=1}^{m} U_i}\right) \\
&= E\left(\prod_{i=1}^{m} e^{tU_i}\right) \\
&= \prod_{i=1}^{m} E(e^{tU_i}) =
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{i=1}^{m} M_{U_i}(t) \\
&= \prod_{i=1}^{m} (1-2t)^{-p_i/2} e^{2\phi_i t/(1-2t)} \\
&= (1-2t)^{-\sum_{i=1}^{m} p_i/2} e^{2\sum_{i=1}^{m} \phi_i t/(1-2t)} \\
&= (1-2t)^{-p/2} e^{2\phi t/(1-2t)}.
\end{aligned}
$$

$\square$

Result 5.8:

$$X \sim N(\mu, 1) \Rightarrow U = X^2 \sim \chi_1^2(\mu^2/2).$$

Proof of Result 5.8:

$$M_U(t) = E(e^{tU}) = E(e^{tX^2})$$
$$= \int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{-1/2(x-\mu)^2 + tx^2} dx.$$

Now the exponent is

$$- 1/2(x^2 - 2\mu x + \mu^2 - 2tx^2)$$
$$= -1/2((1-2t)x^2 - 2\mu x + \mu^2)$$
$$= -1/2\left((1-2t)x^2 - 2\mu x + \mu^2 + \frac{\mu^2}{1-2t} - \frac{\mu^2}{1-2t}\right)$$

$$
\begin{aligned}
&= -1/2\left((1-2t)x^2 - 2\mu x + \frac{\mu^2}{1-2t} + \mu^2 - \frac{\mu^2}{1-2t}\right) \\
&= -1/2\left((1-2t)x^2 - 2\mu x + \frac{\mu^2}{1-2t} + \mu^2\left(1 - \frac{1}{1-2t}\right)\right) \\
&= -1/2\left((1-2t)x^2 - 2\mu x + \frac{\mu^2}{1-2t} + \mu^2\left(\frac{-2t}{1-2t}\right)\right) \\
&= -1/2\left((1-2t)x^2 - 2\mu x + \frac{\mu^2}{1-2t}\right) + \frac{t\mu^2}{1-2t} \\
&= \frac{-1}{2(1-2t)^{-1}}\left(x^2 - 2\frac{\mu}{1-2t}x + \left(\frac{\mu}{1-2t}\right)^2\right) + \frac{t\mu^2}{1-2t} \\
&= \frac{-1}{2(1-2t)^{-1}}\left(x - \frac{\mu}{1-2t}\right)^2 + \frac{t\mu^2}{1-2t}.
\end{aligned}
$$

Thus, $M_U(t)$ is

$$
\begin{aligned}
M_U(t) &= \int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{\frac{-1}{2(1-2t)^{-1}}\left(x-\frac{\mu}{1-2t}\right)^2} e^{\frac{t\mu^2}{1-2t}} dx \\
&= e^{\frac{t\mu^2}{1-2t}} (1-2t)^{-1/2} \int_{-\infty}^{\infty} (2\pi(1-2t)^{-1})^{-1/2} e^{\frac{-1}{2(1-2t)^{-1}}\left(x-\frac{\mu}{1-2t}\right)^2} dx \\
&= (1-2t)^{-1/2} e^{\frac{2(\mu^2/2)t}{1-2t}},
\end{aligned}
$$

which is the MGF of $\chi_1^2(\mu^2/2)$. □

Result 5.9:

$$\boldsymbol{X} \sim N(\boldsymbol{\mu}, \underset{p\times p}{\boldsymbol{I}}) \Rightarrow W = \boldsymbol{X}'\boldsymbol{X} \sim \chi^2_p(\boldsymbol{\mu}'\boldsymbol{\mu}/2).$$

Proof of Result 5.9:

By Result 5.8,

$$X_i^2 \sim \chi_1^2(\mu_i/2) \quad \forall\, i = 1, \ldots, p.$$

By Result 5.4, $X_1, \ldots, X_p$ are mutually independent. Thus $X_1^2, \ldots, X_p^2$ are mutually independent. By Result 5.7,

$$\sum_{i=1}^{p} X_i^2 \sim \chi_p^2 \left(\sum_{i=1}^{p} \mu_i^2/2 \right).$$

Now note that

$$\boldsymbol{X}'\boldsymbol{X} = \sum_{i=1}^{p} X_i^2 \quad \text{and} \quad \boldsymbol{\mu}'\boldsymbol{\mu} = \sum_{i=1}^{p} \mu_i^2.$$

□

Result 5.10:

Suppose $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\underset{p \times p}{\boldsymbol{\Sigma}}$ is nonsingular. Then

$$W = \boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X} \sim \chi^2_p(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}/2).$$

Proof of Result 5.10:

Let $\boldsymbol{Y} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}$. Then

$$E(\boldsymbol{Y}) = \boldsymbol{\Sigma}^{-1/2}E(\boldsymbol{X}) = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu} \quad \text{and}$$
$$\text{Var}(\boldsymbol{Y}) = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1/2} = \boldsymbol{I}.$$

By Result 5.2, $\boldsymbol{Y} \sim N(\boldsymbol{\theta}, \boldsymbol{I})$, where $\boldsymbol{\theta} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}$. By Result 5.9,

$$\boldsymbol{Y}'\boldsymbol{Y} \sim \chi^2_p(\boldsymbol{\theta}'\boldsymbol{\theta}/2).$$

Now note

$$\boldsymbol{Y}'\boldsymbol{Y} = \boldsymbol{X}'\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X} = \boldsymbol{X}'\boldsymbol{\Sigma}^{-1}\boldsymbol{X} \quad \text{and}$$
$$\boldsymbol{\theta}'\boldsymbol{\theta} = \boldsymbol{\mu}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu} = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}.$$

□

Result 5.11:

If $U \sim \chi^2_p(\phi)$, then $\mathbb{P}(U > c)$ is a strictly increasing function of ϕ for fixed p and $c > 0$.

Proof of Result 5.11:

HW problem. $\square$