The Aitken Model

The Aitken Model (AM):

Suppose

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon},$$

where

$$E(\boldsymbol{\varepsilon}) = \boldsymbol{0} \quad \text{and} \quad \text{Var}(\boldsymbol{\varepsilon}) = \sigma^2 \boldsymbol{V}$$

for some $\sigma^2 > 0$ and some known positive definite matrix $\boldsymbol{V}$.

Because $\sigma^2 \boldsymbol{V}$ is a variance matrix, $\boldsymbol{V}$ is symmetric and positive definite,

$\therefore \exists$ a symmetric and positive definite matrix $\boldsymbol{V}^{1/2} \ni$

$$\boldsymbol{V}^{1/2}\boldsymbol{V}^{1/2} = \boldsymbol{V} \quad \text{and} \quad \boldsymbol{V}^{1/2} \text{ is nonsingular with } \boldsymbol{V}^{-1/2} \equiv (\boldsymbol{V}^{1/2})^{-1}.$$

It follows that under the AM,

$$\boldsymbol{V}^{-1/2}\boldsymbol{y} = \boldsymbol{V}^{-1/2}\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{V}^{-1/2}\boldsymbol{\varepsilon} \iff \boldsymbol{z} = \boldsymbol{U}\boldsymbol{\beta} + \boldsymbol{\delta},$$

where

$$\boldsymbol{z} = \boldsymbol{V}^{-1/2}\boldsymbol{y}, \quad \boldsymbol{U} = \boldsymbol{V}^{-1/2}\boldsymbol{X}, \quad \text{and} \quad \boldsymbol{\delta} = \boldsymbol{V}^{-1/2}\boldsymbol{\varepsilon}$$

with

$$E(\boldsymbol{\delta}) = \boldsymbol{0}$$

and

$$\begin{aligned} \text{Var}(\boldsymbol{\delta}) &= \boldsymbol{V}^{-1/2}\sigma^2\boldsymbol{V}\boldsymbol{V}^{-1/2} \\ &= \sigma^2\boldsymbol{V}^{-1/2}\boldsymbol{V}^{1/2}\boldsymbol{V}^{1/2}\boldsymbol{V}^{-1/2} \\ &= \sigma^2\boldsymbol{I}. \end{aligned}$$

Thus, the AM for $\boldsymbol{y}$ is equivalent to the GMM for $\boldsymbol{z} = \boldsymbol{V}^{-1/2}\boldsymbol{y}$.

Estimability in the AM:

The AM is just a special case of the GLM.

Thus, as before, $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable iff $\boldsymbol{c} \in \mathcal{C}(\boldsymbol{X}')$.

Note that

$$\begin{aligned}\mathcal{C}(\boldsymbol{X}') &= \boldsymbol{C}(\boldsymbol{X}'\boldsymbol{V}^{-1/2}) \\ &= \mathcal{C}((\boldsymbol{V}^{-1/2}\boldsymbol{X})') \\ &= \mathcal{C}(\boldsymbol{U}').\end{aligned}$$

Thus, $\boldsymbol{c} \in \mathcal{C}(\boldsymbol{X}') \iff \boldsymbol{c} \in \mathcal{C}(\boldsymbol{U}')$.

Let $\mathcal{L}_{\boldsymbol{y}}$ be the collection of all linear estimators that are linear in $\boldsymbol{y}$.

Let $\mathcal{L}_{\boldsymbol{z}}$ be the collection of all linear estimators in $\boldsymbol{z} = \boldsymbol{V}^{-1/2}\boldsymbol{y}$. Show that

$$\mathcal{L}_{\boldsymbol{y}} = \mathcal{L}_{\boldsymbol{z}}.$$

Estimating $E(\boldsymbol{y})$ under the Aitken Model:

Consider $Q_{\text{GLS}}(\boldsymbol{b}) = (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b})'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b})$.

Finding $\hat{\boldsymbol{\beta}}_{\text{GLS}}$ that minimizes $Q_{\text{GLS}}(\boldsymbol{b})$ over $\boldsymbol{b} \in \mathbb{R}^p$ is a <u>Generalized Least Squares</u> (GLS) problem.

If

$$Q_{\text{GLS}}(\hat{\boldsymbol{\beta}}_{\text{GLS}}) \leq Q_{\text{GLS}}(\boldsymbol{b}) \quad \forall\, \boldsymbol{b} \in \mathbb{R}^p,$$

$\hat{\boldsymbol{\beta}}_{\text{GLS}}$ is a solution to the GLS problem.

$\boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}}$ is known as GLS estimator of $E(\boldsymbol{y})$ if $\hat{\boldsymbol{\beta}}_{\text{GLS}}$ is a solution to the GLS problem.

Show that $\hat{\boldsymbol{\beta}}_{\text{GLS}}$ minimizes $Q_{\text{GLS}}(\boldsymbol{b})$ over $\boldsymbol{b} \in \mathbb{R}^p$ iff $\hat{\boldsymbol{\beta}}_{\text{GLS}}$ solves

$$\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}.$$

These equations are known as the Aitken Equations (AE).

Henceforth, we will use $\hat{\boldsymbol{\beta}}_{\text{GLS}}$ to denote a solution to the AE.

We will use $\hat{\boldsymbol{\beta}}$ or $\hat{\boldsymbol{\beta}}_{\text{OLS}}$ to denote a solution to the NE

$$\boldsymbol{X}'\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{y}. \quad \text{(\underline{O}rdinary \underline{L}east \underline{S}quares)}$$

Because of the equivalence between the AE

$$\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}$$

and the NE

$$\boldsymbol{U}'\boldsymbol{U}\boldsymbol{b} = \boldsymbol{U}'\boldsymbol{z},$$

we know a solution to AE is

$$(\boldsymbol{U}'\boldsymbol{U})^{-}\boldsymbol{U}'\boldsymbol{z} = (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}.$$

Theorem 4.2 (Aitken Theorem):

Suppose the Aitken Model holds. If $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable, then $\boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\text{GLS}}$ is the BLUE of $\boldsymbol{c}'\boldsymbol{\beta}$.

Suppose $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable.

Suppose the AM holds.

Find $\text{Var}(\boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\text{GLS}})$.

Estimation of σ^2 under the Aitken Model:

An unbiased estimator of σ^2 is $\frac{\boldsymbol{z}'(\boldsymbol{I}-\boldsymbol{P_U})\boldsymbol{z}}{n-r}$ based on our previous result for the GMM.

Now, note that

$$\begin{aligned}
\boldsymbol{z}'(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{U}})\boldsymbol{z} &= \boldsymbol{z}'(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{U}})'(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{U}})\boldsymbol{z} \\
&= \|(\boldsymbol{I}-\boldsymbol{P}_{\boldsymbol{U}})\boldsymbol{z}\|^2 = \|\boldsymbol{z}-\boldsymbol{P}_{\boldsymbol{U}}\boldsymbol{z}\|^2 \\
&= \|\boldsymbol{z}-\boldsymbol{U}(\boldsymbol{U}'\boldsymbol{U})^{-}\boldsymbol{U}'\boldsymbol{z}\|^2 \\
&= \|\boldsymbol{z}-\boldsymbol{U}\hat{\boldsymbol{\beta}}_{\text{GLS}}\|^2 = \|\boldsymbol{V}^{-1/2}\boldsymbol{y}-\boldsymbol{V}^{-1/2}\boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}}\|^2 \\
&= \|\boldsymbol{V}^{-1/2}(\boldsymbol{y}-\boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}})\|^2 \\
&= (\boldsymbol{y}-\boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}})'\boldsymbol{V}^{-1}(\boldsymbol{y}-\boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}}).
\end{aligned}$$

Thus,

$$\hat{\sigma}^2_{\text{GLS}} \equiv \frac{(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}})'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}})}{n - r}$$

is an unbiased estimator of σ^2 under the AM.

A Simple Example

Suppose for $i = 1, \ldots, n$,

$$y_i = \beta x_i + \varepsilon_i,$$

where $\varepsilon_1, \ldots, \varepsilon_n$ are uncorrelated, $E(\varepsilon_i) = 0$ and $\text{Var}(\varepsilon_i) = \sigma^2 x_i > 0$.

Find the BLUE of β and an unbiased estimator of σ^2.

Find $\mathrm{Var}(\hat{\beta}_{\mathrm{GLS}})$ for this example.

Find $\hat{\beta}_{\text{OLS}}$ for this example.

Find $\text{Var}(\hat{\beta}_{\text{OLS}})$ in this example.