Idempotency and Projection Matrices

A square matrix $\boldsymbol{P}$ is idempotent iff $\boldsymbol{PP} = \boldsymbol{P}$.

A square matrix $\boldsymbol{P}$ is a projection matrix that projects onto the vector space $\mathcal{S} \subseteq \mathbb{R}^n$ iff

(a) $\boldsymbol{P}$ is idempotent,

(b) $\boldsymbol{Px} \in \mathcal{S} \; \forall \; \boldsymbol{x} \in \mathbb{R}^n$, and

(c) $\boldsymbol{Pz} = \boldsymbol{z} \; \forall \; \boldsymbol{z} \in \mathcal{S}$.

Result P.1:

Suppose $\boldsymbol{P}$ is an idempotent matrix. Prove that $\boldsymbol{P}$ projects onto a vector space $\mathcal{S}$ iff $\mathcal{S} = \mathcal{C}(\boldsymbol{P})$.

Proof of Result P.1:

$(\Longrightarrow)$ Property (b) of a projection matrix implies that

$$\boldsymbol{P}\boldsymbol{x} \in \mathcal{S} \; \forall \; \boldsymbol{x} \; \therefore \mathcal{C}(\boldsymbol{P}) \subseteq \mathcal{S}.$$

By Property (c) of a projection matrix, $\boldsymbol{P}\boldsymbol{z} = \boldsymbol{z} \; \forall \; \boldsymbol{z} \in \mathcal{S}$.

Thus, any $\boldsymbol{z} \in \mathcal{S}$ also in $\mathcal{C}(\boldsymbol{P})$. $\therefore \mathcal{S} \subseteq \mathcal{C}(\boldsymbol{P})$, and we have $\mathcal{C}(\boldsymbol{P}) = \mathcal{S}$.

$(\Longleftarrow)$ Need to show that any idempotent $\boldsymbol{P}$ is a projection matrix that projects onto $\mathcal{C}(\boldsymbol{P})$ as follows:

(a) $\boldsymbol{PP} = \boldsymbol{P}$,

(b) $\boldsymbol{Px} \in \mathcal{C}(\boldsymbol{P}) \; \forall \, \boldsymbol{x}$,

(c) $\boldsymbol{z} \in \mathcal{C}(\boldsymbol{P}) \Rightarrow \exists \, \boldsymbol{x} \ni \boldsymbol{z} = \boldsymbol{Px}$. Therefore, $\boldsymbol{Pz} = \boldsymbol{PPx} = \boldsymbol{Px} = \boldsymbol{z}$. $\square$

Result A.14:

$\boldsymbol{A}\boldsymbol{A}^{-}$ is a projection matrix that projects onto $\mathcal{C}(\boldsymbol{A})$.

Proof of Result A.14:

(a) $(\boldsymbol{AA}^-)(\boldsymbol{AA}^-) = (\boldsymbol{AA}^-\boldsymbol{A})\boldsymbol{A}^- = \boldsymbol{AA}^-$. Therefore, $\boldsymbol{AA}^-$ is idempotent.

(b) $\boldsymbol{AA}^-\boldsymbol{x} = \boldsymbol{Az} \;\forall\; \boldsymbol{x}$, where $\boldsymbol{z} = \boldsymbol{A}^-\boldsymbol{x}$. Thus $\boldsymbol{AA}^-\boldsymbol{x} \in \mathcal{C}(\boldsymbol{A}) \;\forall\; \boldsymbol{x}$.

(c) $\forall\; \boldsymbol{z} \in \mathcal{C}(\boldsymbol{A}), \exists\; \boldsymbol{y} \ni \boldsymbol{z} = \boldsymbol{Ay}, \therefore \boldsymbol{AA}^-\boldsymbol{z} = \boldsymbol{AA}^-\boldsymbol{Ay} = \boldsymbol{Ay} = \boldsymbol{z}$. $\square$

Alternatively, we could have proved idempotency and then shown $\mathcal{C}(\boldsymbol{A}) = \mathcal{C}(\boldsymbol{A}\boldsymbol{A}^-)$ as below:

$$\boldsymbol{A}\boldsymbol{x} = (\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{A})\boldsymbol{x} = (\boldsymbol{A}\boldsymbol{A}^-)\boldsymbol{A}\boldsymbol{x} \Rightarrow \mathcal{C}(\boldsymbol{A}) \subseteq \mathcal{C}(\boldsymbol{A}\boldsymbol{A}^-).$$

$$\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{A}^-\boldsymbol{x}) \Rightarrow \mathcal{C}(\boldsymbol{A}\boldsymbol{A}^-) \subseteq \mathcal{C}(\boldsymbol{A}).$$

$\therefore \mathcal{C}(\boldsymbol{A}) = \mathcal{C}(\boldsymbol{A}\boldsymbol{A}^-)$.

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Result A.15:

$\boldsymbol{I} - \boldsymbol{A}^{-}\boldsymbol{A}$ is a projection matrix that projects onto $\mathcal{N}(\boldsymbol{A})$.

Proof of Result A.15:

(a)

$$\begin{aligned}
&(\boldsymbol{I}-\boldsymbol{A}^{-}\boldsymbol{A})(\boldsymbol{I}-\boldsymbol{A}^{-}\boldsymbol{A}) \\
&=\boldsymbol{I}-\boldsymbol{A}^{-}\boldsymbol{A}-\boldsymbol{A}^{-}\boldsymbol{A}+\boldsymbol{A}^{-}\boldsymbol{A}\boldsymbol{A}^{-}\boldsymbol{A} \\
&=\boldsymbol{I}-\boldsymbol{A}^{-}\boldsymbol{A}-\boldsymbol{A}^{-}\boldsymbol{A}+\boldsymbol{A}^{-}\boldsymbol{A} \\
&=\boldsymbol{I}-\boldsymbol{A}^{-}\boldsymbol{A}.
\end{aligned}$$

(b) Note that

$$
\begin{aligned}
\boldsymbol{A}(\boldsymbol{I}-\boldsymbol{A}^{-}\boldsymbol{A})\boldsymbol{x} &= (\boldsymbol{A}-\boldsymbol{A}\boldsymbol{A}^{-}\boldsymbol{A})\boldsymbol{x} \\
&= (\boldsymbol{A}-\boldsymbol{A})\boldsymbol{x} \\
&= \boldsymbol{0}\ \forall\ \boldsymbol{x}. \\
&\therefore (\boldsymbol{I}-\boldsymbol{A}^{-}\boldsymbol{A})\boldsymbol{x} \in \mathcal{N}(\boldsymbol{A})\ \forall\ \boldsymbol{x}.
\end{aligned}
$$

(c) If $\boldsymbol{z} \in \mathcal{N}(\boldsymbol{A})$, then

$$\begin{aligned}(\boldsymbol{I} - \boldsymbol{A}^{-}\boldsymbol{A})\boldsymbol{z} &= \boldsymbol{z} - \boldsymbol{A}^{-}\boldsymbol{A}\boldsymbol{z} \\ &= \boldsymbol{z} - \boldsymbol{0} \\ &= \boldsymbol{z}.\end{aligned}$$

□

Prove that $\mathcal{C}(\boldsymbol{I} - \boldsymbol{A}^{-}\boldsymbol{A}) = \mathcal{N}(\boldsymbol{A})$.

Proof:

The result follows from Result A.15 and P.1.

An alternative proof is as follows.

Proof:

Suppose $\boldsymbol{z} \in \mathcal{N}(\boldsymbol{A})$. Then

$$\begin{aligned} \boldsymbol{A}\boldsymbol{z} = \boldsymbol{0} &\Rightarrow \boldsymbol{A}^{-}\boldsymbol{A}\boldsymbol{z} = \boldsymbol{0} \\ &\Rightarrow \boldsymbol{z} - \boldsymbol{A}^{-}\boldsymbol{A}\boldsymbol{z} = \boldsymbol{z} \\ &\Rightarrow (\boldsymbol{I} - \boldsymbol{A}^{-}\boldsymbol{A})\boldsymbol{z} = \boldsymbol{z} \\ &\Rightarrow \boldsymbol{z} \in \mathcal{C}(\boldsymbol{I} - \boldsymbol{A}^{-}\boldsymbol{A}). \end{aligned}$$

$\therefore \mathcal{N}(\boldsymbol{A}) \subseteq \mathcal{C}(\boldsymbol{I} - \boldsymbol{A}^{-}\boldsymbol{A})$.

Suppose $z \in \mathcal{C}(\boldsymbol{I} - \boldsymbol{A}^{-}\boldsymbol{A})$. Then $\exists\, \boldsymbol{x} \ni z = (\boldsymbol{I} - \boldsymbol{A}^{-}\boldsymbol{A})\boldsymbol{x}$. Thus

$$\begin{aligned} \boldsymbol{A}z &= \boldsymbol{A}(\boldsymbol{I} - \boldsymbol{A}^{-}\boldsymbol{A})\boldsymbol{x} \\ &= (\boldsymbol{A} - \boldsymbol{A}\boldsymbol{A}^{-}\boldsymbol{A})\boldsymbol{x} \\ &= (\boldsymbol{A} - \boldsymbol{A})\boldsymbol{x} \\ &= \boldsymbol{0}. \end{aligned}$$

Thus, $z \in \mathcal{N}(\boldsymbol{A})$. It follows that $\mathcal{C}(\boldsymbol{I} - \boldsymbol{A}^{-}\boldsymbol{A}) \subseteq \mathcal{N}(\boldsymbol{A})$. Hence, $\mathcal{C}(\boldsymbol{I} - \boldsymbol{A}^{-}\boldsymbol{A}) = \mathcal{N}(\boldsymbol{A})$. $\square$

Result A.16:

Any symmetric and idempotent matrix $\boldsymbol{P}$ is the unique symmetric projection matrix that projects onto $\mathcal{C}(\boldsymbol{P})$.

Proof of Result A.16:

Suppose $\boldsymbol{Q}$ is a symmetric projection matrix that projects onto $\mathcal{C}(\boldsymbol{P})$. Then

$$\begin{aligned} &\boldsymbol{Pz} = \boldsymbol{Qz} = \boldsymbol{z} \ \forall \ \boldsymbol{z} \in \mathcal{C}(\boldsymbol{P}) \\ &\Rightarrow \boldsymbol{PPx} = \boldsymbol{QPx} \ \forall \ \boldsymbol{x} \\ &\Rightarrow \boldsymbol{Px} = \boldsymbol{QPx} \ \forall \ \boldsymbol{x} \\ &\Rightarrow \boldsymbol{P} = \boldsymbol{QP}. \end{aligned}$$

Now $\boldsymbol{Q}$ is a projection matrix that projects on $\mathcal{C}(\boldsymbol{P})$, therefore, $\mathcal{C}(\boldsymbol{P}) = \mathcal{C}(\boldsymbol{Q})$. Thus

$$\begin{aligned} \boldsymbol{Qz} = \boldsymbol{Pz} = \boldsymbol{z} \;\forall\; \boldsymbol{z} \in \mathcal{C}(\boldsymbol{Q}) \\ \Rightarrow \boldsymbol{QQx} = \boldsymbol{PQx} \;\forall\; \boldsymbol{x} \\ \Rightarrow \boldsymbol{Qx} = \boldsymbol{PQx} \;\forall\; \boldsymbol{x} \\ \Rightarrow \boldsymbol{Q} = \boldsymbol{PQ}. \end{aligned}$$

Now note that

$$\begin{aligned}(\boldsymbol{P}-\boldsymbol{Q})'(\boldsymbol{P}-\boldsymbol{Q}) &= \boldsymbol{P}'\boldsymbol{P}-\boldsymbol{P}'\boldsymbol{Q}-\boldsymbol{Q}'\boldsymbol{P}+\boldsymbol{Q}'\boldsymbol{Q} \\ &= \boldsymbol{P}\boldsymbol{P}-\boldsymbol{P}\boldsymbol{Q}-\boldsymbol{Q}\boldsymbol{P}+\boldsymbol{Q}\boldsymbol{Q} \\ &= \boldsymbol{P}-\boldsymbol{Q}-\boldsymbol{P}+\boldsymbol{Q} \\ &= \boldsymbol{0}.\end{aligned}$$

$\therefore \boldsymbol{P}-\boldsymbol{Q}=\boldsymbol{0} \Rightarrow \boldsymbol{P}=\boldsymbol{Q}.$ □

Any symmetric, idempotent matrix $\boldsymbol{P}$ is known as an orthogonal projection matrix because $(\boldsymbol{P}\boldsymbol{x}) \perp (\boldsymbol{x} - \boldsymbol{P}\boldsymbol{x})$, i.e.,

$$\begin{aligned}
(\boldsymbol{P}\boldsymbol{x})'(\boldsymbol{x} - \boldsymbol{P}\boldsymbol{x}) &= \boldsymbol{x}'\boldsymbol{P}\boldsymbol{x} - \boldsymbol{x}'\boldsymbol{P}'\boldsymbol{P}\boldsymbol{x} \\
&= \boldsymbol{x}'\boldsymbol{P}\boldsymbol{x} - \boldsymbol{x}'\boldsymbol{P}\boldsymbol{P}\boldsymbol{x} \\
&= \boldsymbol{x}'\boldsymbol{P}\boldsymbol{x} - \boldsymbol{x}'\boldsymbol{P}\boldsymbol{x} \\
&= 0.
\end{aligned}$$

Corollary A.4:

If $\boldsymbol{P}$ is a symmetric projection matrix, then $\boldsymbol{I} - \boldsymbol{P}$ is a symmetric projection matrix that projects onto $\mathcal{C}(\boldsymbol{P})^{\perp} = \mathcal{N}(\boldsymbol{P})$.

Proof of Corollary A.4:

First note that $\mathcal{C}(\boldsymbol{P})^{\perp} = \mathcal{N}(\boldsymbol{P}') = \mathcal{N}(\boldsymbol{P})$ by the symmetry of $\boldsymbol{P}$.

We need to show that properties (a-c) of a projection matrix hold for $\boldsymbol{I} - \boldsymbol{P}$ onto $\mathcal{N}(\boldsymbol{P})$.

(a) Is $\boldsymbol{I} - \boldsymbol{P}$ idempotent?

$$\begin{aligned}(\boldsymbol{I} - \boldsymbol{P})(\boldsymbol{I} - \boldsymbol{P}) &= \boldsymbol{I} - \boldsymbol{P} - \boldsymbol{P} + \boldsymbol{P}\boldsymbol{P} \\ &= \boldsymbol{I} - \boldsymbol{P} - \boldsymbol{P} + \boldsymbol{P} \\ &= \boldsymbol{I} - \boldsymbol{P}.\end{aligned}$$

(b) Is $(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{x} \in \mathcal{N}(\boldsymbol{P}) \;\forall\; \boldsymbol{x}$?

$$\begin{aligned} \boldsymbol{P}(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{x} &= (\boldsymbol{P} - \boldsymbol{P}\boldsymbol{P})\boldsymbol{x} \\ &= (\boldsymbol{P} - \boldsymbol{P})\boldsymbol{x} \\ &= \boldsymbol{0}. \end{aligned}$$

$\therefore (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{x} \in \mathcal{N}(\boldsymbol{P}) \;\forall\; \boldsymbol{x}$.

(c) Does $(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{z}=\boldsymbol{z}\ \forall\ \boldsymbol{z}\in\mathcal{N}(\boldsymbol{P})$?

$$\begin{aligned}\forall\ \boldsymbol{z}\in\mathcal{N}(\boldsymbol{P}),\ (\boldsymbol{I}-\boldsymbol{P})\boldsymbol{z} &= \boldsymbol{z}-\boldsymbol{P}\boldsymbol{z}\\ &= \boldsymbol{z}-\boldsymbol{0}\\ &= \boldsymbol{z}.\end{aligned}$$

Finally, we should note that $(\boldsymbol{I}-\boldsymbol{P})'=\boldsymbol{I}'-\boldsymbol{P}'=\boldsymbol{I}-\boldsymbol{P}$ so that $\boldsymbol{I}-\boldsymbol{P}$ is symmetric as claimed in statement of the result. □

Suppose $\boldsymbol{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

- Find the orthogonal projection matrix that projects onto $\mathcal{C}(\boldsymbol{A})$.
- Find the orthogonal projection matrix that projects onto $\mathcal{N}(\boldsymbol{A}')$.
- Find the orthogonal projection of $\boldsymbol{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ onto $\mathcal{C}(\boldsymbol{A})$ and onto $\mathcal{N}(\boldsymbol{A}')$.

Need to find a symmetric, idempotent matrix whose column space is $\mathcal{C}(\boldsymbol{A})$, where

$$\mathcal{C}(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 = x_2\}.$$

Thus, $\boldsymbol{P}$ must have the form

$$\boldsymbol{P} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}.$$

Because $\boldsymbol{P}$ must be idempotent,

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} a & a \\ a & a \end{bmatrix} = \begin{bmatrix} 2a^2 & 2a^2 \\ 2a^2 & 2a^2 \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}.$$

This implies $2a^2 = a \Rightarrow a = 1/2. \therefore \boldsymbol{P} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$

We know

$$\boldsymbol{I} - \boldsymbol{P} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

is the orthogonal projection matrix that projects onto $\mathcal{C}(\boldsymbol{P})^{\perp} = \mathcal{C}(\boldsymbol{A})^{\perp} = \mathcal{N}(\boldsymbol{A}')$.

$$\boldsymbol{P}\begin{bmatrix}4\\2\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix},(\boldsymbol{I}-\boldsymbol{P})\begin{bmatrix}4\\2\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}.$$