Miscellaneous Results, Solving Equations, and Generalized Inverses

Result A.7:

Suppose $\mathcal{S}$ and $\mathcal{T}$ are vector spaces. If $\mathcal{S} \subseteq \mathcal{T}$ and $dim(\mathcal{S}) = dim(\mathcal{T})$, then $\mathcal{S} = \mathcal{T}$.

Result A.8:

Suppose $\underset{m\times n}{\boldsymbol{A}}$ and $\underset{m\times 1}{\boldsymbol{b}}$ satisfy

$$\underset{m\times n}{\boldsymbol{A}}\boldsymbol{x} + \underset{m\times 1}{\boldsymbol{b}} = \boldsymbol{0} \;\forall\; \boldsymbol{x} \in \mathbb{R}^n.$$

Then $\underset{m\times n}{\boldsymbol{A}} = \underset{m\times n}{\boldsymbol{0}}$ and $\underset{m\times 1}{\boldsymbol{b}} = \underset{m\times 1}{\boldsymbol{0}}$.

Corollary A.1:

If $\underset{m\times n}{\boldsymbol{B}}$ and $\underset{m\times n}{\boldsymbol{C}}$ satisfy $\boldsymbol{Bx} = \boldsymbol{Cx}\ \forall\ \boldsymbol{x} \in \mathbb{R}^n$, then $\boldsymbol{B} = \boldsymbol{C}$.

Corollary A.2:

Suppose $\underset{m\times n}{\boldsymbol{A}}$ has full column rank. Then

$$\boldsymbol{AB} = \boldsymbol{AC} \Longrightarrow \boldsymbol{B} = \boldsymbol{C}.$$

□

Lemma A.1:

$$\boldsymbol{C}'\boldsymbol{C} = \boldsymbol{0} \Longrightarrow \boldsymbol{C} = \boldsymbol{0}.$$

Proof of Lemma A.1:

- Let $\boldsymbol{c}_i$ denote the i^{th} column of $\boldsymbol{C}$. Then the i^{th} diagonal element of $\boldsymbol{C}'\boldsymbol{C}$ is $\boldsymbol{c}_i'\boldsymbol{c}_i$.
- $\because \boldsymbol{C}'\boldsymbol{C} = \boldsymbol{0}, \boldsymbol{c}_i'\boldsymbol{c}_i = 0 \;\forall\; i = 1, \ldots, n$.
- Now $\boldsymbol{c}_i'\boldsymbol{c}_i = 0 \;\forall\; i \Longrightarrow \boldsymbol{c}_i = \boldsymbol{0} \;\forall\; i \Longrightarrow \boldsymbol{C} = \boldsymbol{0}$. $\square$

Another Result on the Rank of a Product

Suppose $rank(\underset{m\times n}{\boldsymbol{A}}) = n$ and $rank(\underset{k\times l}{\boldsymbol{C}}) = k$. Then

$$rank(\underset{n\times k}{\boldsymbol{B}}) = rank(\boldsymbol{ABC}).$$

<u>Proof:</u> HW problem. □

<u>Corollaries:</u>

- If $\boldsymbol{A}$ is full-column rank, $rank(\boldsymbol{AB}) = rank(\boldsymbol{B})$.
- If $\boldsymbol{C}$ is full-row rank, $rank(\boldsymbol{BC}) = rank(\boldsymbol{B})$.

Solving Equations:

Consider a system of linear equations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{c},$$

where $\underset{m\times n}{\boldsymbol{A}}$ is a known matrix and $\underset{m\times 1}{\boldsymbol{c}}$ is a known vector.

- We seek a solution vector $\boldsymbol{x}$ that satisfies $\boldsymbol{Ax} = \boldsymbol{c}$.

- If $m = n$ so that $\underset{m \times n}{\boldsymbol{A}}$ is a square and if $\underset{m \times n}{\boldsymbol{A}}$ is nonsingular, then

$$\boldsymbol{A}^{-1}\boldsymbol{Ax} = \boldsymbol{A}^{-1}\boldsymbol{c} \Longrightarrow \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{c}$$

  is the unique solution to $\boldsymbol{Ax} = \boldsymbol{c}$.

- If $\underset{m\times n}{\boldsymbol{A}}$ is singular or not square, $\boldsymbol{Ax} = \boldsymbol{c}$ may have no solution or infinitely many solutions or a unique solution.

- A system of equations $\boldsymbol{Ax} = \boldsymbol{c}$ is consistent if there exists a solution $\boldsymbol{x}^*$ such that $\boldsymbol{Ax}^* = \boldsymbol{c}$.

- A systems of equation $\boldsymbol{Ax} = \boldsymbol{c}$ is inconsistent if $\boldsymbol{Ax} \neq \boldsymbol{c} \; \forall \; \boldsymbol{x} \in \mathbb{R}^n$.

Result A.9:

A system of equations $\boldsymbol{Ax} = \boldsymbol{c}$ is consistent iff $\boldsymbol{c} \in \mathcal{C}(\boldsymbol{A})$.

- Provide an example $\underset{m\times n}{\boldsymbol{A}}, \underset{m\times 1}{\boldsymbol{c}} \ni \boldsymbol{Ax} = \boldsymbol{c}$ is inconsistent.
- Provide an example $\underset{m\times n}{\boldsymbol{A}}, \underset{m\times 1}{\boldsymbol{c}} \ni \boldsymbol{Ax} = \boldsymbol{c}$ is consistent.

Generalized Inverse

- A matrix $\boldsymbol{G}$ is a generalized inverse (GI) of a matrix $\boldsymbol{A}$ iff

$$\boldsymbol{AGA} = \boldsymbol{A}.$$

- Every matrix has at least one GI. We will use $\boldsymbol{A}^-$ to denote a GI of a matrix $\boldsymbol{A}$.

If $\boldsymbol{A}$ is nonsingular, then $\boldsymbol{A}^{-1}$ is the unique GI of $\boldsymbol{A}$.

Result A.10:

Suppose $rank(\underset{m\times n}{\boldsymbol{A}}) = r$. If $\boldsymbol{A}$ can be partitioned as

$$\boldsymbol{A} = \begin{bmatrix} \underset{r\times r}{\boldsymbol{C}} & \underset{r\times(n-r)}{\boldsymbol{D}} \\ \underset{(m-r)\times r}{\boldsymbol{E}} & \underset{(m-r)\times(n-r)}{\boldsymbol{F}} \end{bmatrix},$$

where $rank(\underset{r\times r}{\boldsymbol{C}}) = r$, then

$$\underset{n\times m}{\boldsymbol{G}} = \begin{bmatrix} \boldsymbol{C}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

is a GI of $\underset{m\times n}{\boldsymbol{A}}$.

Proof of Result A.10:

$$\boldsymbol{AG} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{F} \end{bmatrix} \begin{bmatrix} \boldsymbol{C}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{EC}^{-1} & \boldsymbol{0} \end{bmatrix}.$$

$$\therefore \boldsymbol{AGA} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{EC}^{-1} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{F} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{EC}^{-1}\boldsymbol{D} \end{bmatrix}.$$

We need to show $\boldsymbol{EC}^{-1}\boldsymbol{D} = \boldsymbol{F}$.

First note that

$$rank(\underset{r\times r}{\boldsymbol{C}}) = r \Longrightarrow rank([\boldsymbol{C}, \boldsymbol{D}]) = r$$

$\because$ $[\boldsymbol{C}, \boldsymbol{D}]$ has at least r LI columns and at most r LI rows.

$(rank([\boldsymbol{C}, \boldsymbol{D}]) \geq r$ and $rank([\boldsymbol{C}, \boldsymbol{D}]) \leq r \Longrightarrow rank([\boldsymbol{C}, \boldsymbol{D}]) = r.)$

Now $rank\left(\begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{F} \end{bmatrix}\right) = r \Longrightarrow$ each row of $[\boldsymbol{E}, \boldsymbol{F}]$ is a LC of the rows of $[\boldsymbol{C}, \boldsymbol{D}]$.

Thus $\exists$ a matrix $\boldsymbol{K} \ni$

$$\begin{aligned} &\boldsymbol{K}[\boldsymbol{C}, \boldsymbol{D}] = [\boldsymbol{E}, \boldsymbol{F}] \\ \Longleftrightarrow\ &[\boldsymbol{KC}, \boldsymbol{KD}] = [\boldsymbol{E}, \boldsymbol{F}] \\ \Longleftrightarrow\ &\boldsymbol{KC} = \boldsymbol{E}, \boldsymbol{KD} = \boldsymbol{F}. \end{aligned}$$

Now $\boldsymbol{KC} = \boldsymbol{E} \Longleftrightarrow \boldsymbol{K} = \boldsymbol{EC}^{-1}$. Together with $\boldsymbol{KD} = \boldsymbol{F}$, this implies $\boldsymbol{EC}^{-1}\boldsymbol{D} = \boldsymbol{F}$. □

Permutation Matrix

A matrix $\underset{n\times n}{\boldsymbol{P}}$ is a permutation matrix if the rows of $\boldsymbol{P}$ are the same as the rows of $\underset{n\times n}{\boldsymbol{I}}$ but not necessarily in the same order.

Example:

- $\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
- If $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$, then $\boldsymbol{PA} = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 9 & 10 & 11 & 12 \end{bmatrix}$.
- Order of rows of $\boldsymbol{PA}$ are permuted relative to order of rows of $\boldsymbol{A}$.

Example:

- $\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$
- Then $\boldsymbol{BP} = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \end{bmatrix}.$
- Order of columns of $\boldsymbol{BP}$ are permuted relative to order of columns of $\boldsymbol{B}$.

A Permutation Matrix is Nonsingular

The rows (and columns) of a permutation matrix are the same as those of the identity matrix. Thus, a permutation matrix has full rank and is therefore nonsingular.

Furthermore, if $\boldsymbol{P} = [\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n]$ is a permutation matrix,

$$\boldsymbol{P}^{-1} = \boldsymbol{P}' \because \boldsymbol{p}_i'\boldsymbol{p}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Result A.11:

Suppose $rank(\underset{m\times n}{\boldsymbol{A}}) = r$. There exist permutation matrices $\underset{m\times m}{\boldsymbol{P}}$ and $\underset{n\times n}{\boldsymbol{Q}}$ such that

$$\boldsymbol{PAQ} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{F} \end{bmatrix},$$

where $rank(\underset{r\times r}{\boldsymbol{C}}) = r$. Furthermore,

$$\boldsymbol{G} = \boldsymbol{Q} \begin{bmatrix} \boldsymbol{C}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \boldsymbol{P}$$

is a GI of $\boldsymbol{A}$.

Proof of Result A.11:

- Because $rank(\boldsymbol{A}) = r$, there exists a set of r rows of $\boldsymbol{A}$ that are LI.
- Let $\boldsymbol{P}$ be a permutation matrix, $\ni$ the first r rows of $\boldsymbol{PA}$ are LI.
- Let $\boldsymbol{H}$ be the matrix consisting of the first r rows of $\boldsymbol{PA}$. Then $rank(\boldsymbol{H}) = r$.

- This implies that $\exists$ a set of r columns of $\boldsymbol{H}$ that are LI.
- Let $\boldsymbol{Q}$ be a permutation matrix $\ni$ the first r columns of $\boldsymbol{HQ}$ are LI.
- Then the submatrix consisting of the first r rows and first r columns of $\boldsymbol{PAQ}$ has rank r.

- Thus we can partition $\boldsymbol{PAQ}$ as $\begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{F} \end{bmatrix}$, where $rank(\underset{r\times r}{\boldsymbol{C}}) = r$.

- By Result A.10, $\begin{bmatrix} \boldsymbol{C}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$ is a GI for $\boldsymbol{PAQ} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{F} \end{bmatrix}$.

$$PAQ = \begin{bmatrix} C & D \\ E & F \end{bmatrix} \iff A = P^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} Q^{-1}$$

$$\therefore AQ \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} PA =$$

$$= P^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} Q^{-1} Q \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} PP^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} Q^{-1}$$

$$= P^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C & D \\ E & F \end{bmatrix} Q^{-1}$$

$$= P^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} Q^{-1} = A.$$

$\square$

Use Result A.11 to find a GI for

$$A = \begin{bmatrix} 1 & 1 & 3 & 4 \\ 2 & 2 & 6 & 8 \\ 3 & 3 & 0 & 12 \end{bmatrix}.$$

Algorithm for finding a GI of $\boldsymbol{A}$:

1. Find an $r \times r$ nonsingular submatrix of $\boldsymbol{A}$, where $r = rank(\boldsymbol{A})$. Call this matrix $\boldsymbol{W}$.

2. Compute $(\boldsymbol{W}^{-1})'$.

3. Replace each element of $\boldsymbol{W}$ in $\boldsymbol{A}$ with the corresponding elements of $(\boldsymbol{W}^{-1})'$.

4. Replace all other elements in $\boldsymbol{A}$ with zeros.

5. Transpose resulting matrix to get $\boldsymbol{A}^-$.

Result A.12:

Let $\boldsymbol{Ax} = \boldsymbol{c}$ be a consistent system of equations, and let $\boldsymbol{G}$ be any GI of $\boldsymbol{A}$. Then $\boldsymbol{Gc}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{c}$, i.e., $\boldsymbol{AGc} = \boldsymbol{c}$.

Result A.13:

Let $\boldsymbol{Ax} = \boldsymbol{c}$ be a consistent system of equations, and let $\boldsymbol{G}$ be any GI of $\boldsymbol{A}$. Then $\tilde{\boldsymbol{x}}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{c}$ iff $\exists\, \boldsymbol{z} \ni \tilde{\boldsymbol{x}} = \boldsymbol{Gc} + (\boldsymbol{I} - \boldsymbol{GA})\boldsymbol{z}$.

Prove that a consistent system of equations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}$$

has a unique solution if and only if $\boldsymbol{A}$ is full-column rank and infinitely many solutions if and only if $\boldsymbol{A}$ is less than full-column rank.