Miscellaneous Results, Solving Equations, and Generalized Inverses

Result A.7:

Suppose $\mathcal{S}$ and $\mathcal{T}$ are vector spaces. If $\mathcal{S} \subseteq \mathcal{T}$ and $dim(\mathcal{S}) = dim(\mathcal{T})$, then $\mathcal{S} = \mathcal{T}$.

Proof of Result A.7:

- Let k denote the common dimension of $\mathcal{S}$ and $\mathcal{T}$. Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$ be a basis for $\mathcal{S}$.
- Then $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$ are LI vectors in $\mathcal{T}$, and are thus a basis for $\mathcal{T}$ by V4. Because $\mathcal{S}$ and $\mathcal{T}$ have a common basis, $\mathcal{S} = \mathcal{T}$. $\square$

Result A.8:

Suppose $\underset{m\times n}{\boldsymbol{A}}$ and $\underset{m\times 1}{\boldsymbol{b}}$ satisfy

$$\underset{m\times n}{\boldsymbol{A}}\boldsymbol{x} + \underset{m\times 1}{\boldsymbol{b}} = \boldsymbol{0} \ \forall \ \boldsymbol{x} \in \mathbb{R}^n.$$

Then $\underset{m\times n}{\boldsymbol{A}} = \underset{m\times n}{\boldsymbol{0}}$ and $\underset{m\times 1}{\boldsymbol{b}} = \underset{m\times 1}{\boldsymbol{0}}$.

Proof of Result A.8:

- When $\boldsymbol{x} = \boldsymbol{0}$, $\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} = \boldsymbol{0}$ becomes $\boldsymbol{A}\boldsymbol{0} + \boldsymbol{b} = \boldsymbol{0} \Longrightarrow \boldsymbol{b} = \boldsymbol{0}$.

- Next, let columns of $\boldsymbol{A}$ be $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \ni \boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]$.
  When $\boldsymbol{x} = \boldsymbol{e}_i, \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ becomes
  $[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_i, \ldots, \boldsymbol{a}_n]\boldsymbol{e}_i = \boldsymbol{0} \Longrightarrow \boldsymbol{a}_i = \boldsymbol{0}$.
  This is true for all $i = 1, \ldots, n$. $\therefore \boldsymbol{A} = \boldsymbol{0}$.

- Alternatively, note that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \; \forall \; \boldsymbol{x} \in \mathbb{R}^n \Longrightarrow \mathcal{C}(\boldsymbol{A}) = \{\boldsymbol{0}\} \Longrightarrow \boldsymbol{A} = \boldsymbol{0}$.

Corollary A.1:

If $\underset{m \times n}{\boldsymbol{B}}$ and $\underset{m \times n}{\boldsymbol{C}}$ satisfy $\boldsymbol{Bx} = \boldsymbol{Cx} \; \forall \; \boldsymbol{x} \in \mathbb{R}^n$, then $\boldsymbol{B} = \boldsymbol{C}$.

Proof of Corollary A.1:

$$\boldsymbol{Bx} = \boldsymbol{Cx} \; \forall \; \boldsymbol{x} \in \mathbb{R}^n$$

$$\implies \boldsymbol{Bx} - \boldsymbol{Cx} = \boldsymbol{0} \; \forall \; \boldsymbol{x} \in \mathbb{R}^n$$

$$\implies (\boldsymbol{B} - \boldsymbol{C})\boldsymbol{x} = \boldsymbol{0} \; \forall \; \boldsymbol{x} \in \mathbb{R}^n$$

$$\implies \boldsymbol{B} - \boldsymbol{C} = \boldsymbol{0} \text{ by Result A.8}$$

$$\therefore \boldsymbol{B} = \boldsymbol{C}.$$

□

Corollary A.2:

Suppose $\underset{m \times n}{A}$ has full column rank. Then

$$AB = AC \Longrightarrow B = C.$$

□

Proof of Corollary A.2:

- Result A.3 implies that $\mathcal{N}(\boldsymbol{A}) = \{\mathbf{0}\}$. Thus, $\boldsymbol{A}\boldsymbol{x} = \mathbf{0} \Longrightarrow \boldsymbol{x} = \mathbf{0}$.
- Now

$$
\begin{aligned}
\boldsymbol{AB} = \boldsymbol{AC} &\Longrightarrow \boldsymbol{AB} - \boldsymbol{AC} = \mathbf{0} \\
&\Longrightarrow \boldsymbol{A}(\boldsymbol{B} - \boldsymbol{C}) = \mathbf{0} \\
&\Longrightarrow \boldsymbol{A} \text{ times each column of } \boldsymbol{B} - \boldsymbol{C} \text{ is } \mathbf{0} \\
&\Longrightarrow \text{ Each column of } \boldsymbol{B} - \boldsymbol{C} \text{ is } \mathbf{0} \\
&\Longrightarrow \boldsymbol{B} - \boldsymbol{C} = \mathbf{0} \Longrightarrow \boldsymbol{B} = \boldsymbol{C}.
\end{aligned}
$$

□

Lemma A.1:

$$\boldsymbol{C}'\boldsymbol{C} = \mathbf{0} \Longrightarrow \boldsymbol{C} = \mathbf{0}.$$

Proof of Lemma A.1:

- Let $\boldsymbol{c}_i$ denote the i^{th} column of $\boldsymbol{C}$. Then the i^{th} diagonal element of $\boldsymbol{C}'\boldsymbol{C}$ is $\boldsymbol{c}_i'\boldsymbol{c}_i$.
- $\because \boldsymbol{C}'\boldsymbol{C} = \mathbf{0}, \boldsymbol{c}_i'\boldsymbol{c}_i = 0 \ \forall \ i = 1, \ldots, n$.
- Now $\boldsymbol{c}_i'\boldsymbol{c}_i = 0 \ \forall \ i \Longrightarrow \boldsymbol{c}_i = \mathbf{0} \ \forall \ i \Longrightarrow \boldsymbol{C} = \mathbf{0}$. $\square$

Another Result on the Rank of a Product

Suppose $rank(\underset{m\times n}{\boldsymbol{A}}) = n$ and $rank(\underset{k\times l}{\boldsymbol{C}}) = k$. Then

$$rank(\underset{n\times k}{\boldsymbol{B}}) = rank(\boldsymbol{ABC}).$$

<u>Proof:</u> HW problem. □

<u>Corollaries:</u>

- If $\boldsymbol{A}$ is full-column rank, $rank(\boldsymbol{AB}) = rank(\boldsymbol{B})$.
- If $\boldsymbol{C}$ is full-row rank, $rank(\boldsymbol{BC}) = rank(\boldsymbol{B})$.

Solving Equations:

Consider a system of linear equations

$$\boldsymbol{Ax} = \boldsymbol{c},$$

where $\underset{m\times n}{\boldsymbol{A}}$ is a known matrix and $\underset{m\times 1}{\boldsymbol{c}}$ is a known vector.

- We seek a solution vector $\boldsymbol{x}$ that satisfies $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}$.

- If $m = n$ so that $\underset{m \times n}{\boldsymbol{A}}$ is a square and if $\underset{m \times n}{\boldsymbol{A}}$ is nonsingular, then

$$\boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{c} \Longrightarrow \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{c}$$

  is the unique solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}$.

- If $\underset{m\times n}{A}$ is singular or not square, $Ax = c$ may have no solution or infinitely many solutions or a unique solution.

- A system of equations $\boldsymbol{Ax} = \boldsymbol{c}$ is <u>consistent</u> if there exists a solution $\boldsymbol{x}^*$ such that $\boldsymbol{Ax}^* = \boldsymbol{c}$.

- A systems of equation $\boldsymbol{Ax} = \boldsymbol{c}$ is <u>inconsistent</u> if $\boldsymbol{Ax} \neq \boldsymbol{c} \ \forall \ \boldsymbol{x} \in \mathbb{R}^n$.

Result A.9:

A system of equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}$ is consistent iff $\boldsymbol{c} \in \mathcal{C}(\boldsymbol{A})$.

- Provide an example $\underset{m \times n}{\boldsymbol{A}}, \underset{m \times 1}{\boldsymbol{c}} \ni \boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}$ is inconsistent.
- Provide an example $\underset{m \times n}{\boldsymbol{A}}, \underset{m \times 1}{\boldsymbol{c}} \ni \boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}$ is consistent.

- $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then $\boldsymbol{c} \notin \mathcal{C}(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^3 : x_3 = 0\}$. Thus $\boldsymbol{Ax} = \boldsymbol{c}$ is inconsistent.

- $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$. Here $\boldsymbol{Ax} = \boldsymbol{c}$ is consistent because $\boldsymbol{c} \in \mathcal{C}(\boldsymbol{A})$. The unique solution to $\boldsymbol{Ax} = \boldsymbol{c}$ is $\boldsymbol{x}^* = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

- $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} 7 \\ 3 \\ 3 \end{bmatrix}.$

- Here $\boldsymbol{c} \in \mathcal{C}(\boldsymbol{A})$; hence, $\boldsymbol{Ax} = \boldsymbol{c}$ is consistent.

- There are infinitely many solution given by

$$\{\boldsymbol{x} \in \mathbb{R}^3 : x_1 + 3x_3 = 7, x_2 + 3x_3 = 3\}.$$

Generalized Inverse

- A matrix $\boldsymbol{G}$ is a generalized inverse (GI) of a matrix $\boldsymbol{A}$ iff

$$\boldsymbol{AGA} = \boldsymbol{A}.$$

- Every matrix has at least one GI. We will use $\boldsymbol{A}^-$ to denote a GI of a matrix $\boldsymbol{A}$.

If $\boldsymbol{A}$ is nonsingular, then $\boldsymbol{A}^{-1}$ is the unique GI of $\boldsymbol{A}$.

Proof:

$$\boldsymbol{A}\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}\boldsymbol{A} = \boldsymbol{A}.$$

$$\begin{aligned}\boldsymbol{A}\boldsymbol{G}\boldsymbol{A} = \boldsymbol{A} &\Longrightarrow \boldsymbol{A}^{-1}(\boldsymbol{A}\boldsymbol{G}\boldsymbol{A})\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{A}^{-1} \\ &\Longrightarrow \boldsymbol{G} = \boldsymbol{A}^{-1}.\end{aligned}$$

$\square$

Result A.10:

Suppose $rank(\underset{m\times n}{\boldsymbol{A}}) = r$. If $\boldsymbol{A}$ can be partitioned as

$$\boldsymbol{A} = \begin{bmatrix} \underset{r\times r}{\boldsymbol{C}} & \underset{r\times(n-r)}{\boldsymbol{D}} \\ \underset{(m-r)\times r}{\boldsymbol{E}} & \underset{(m-r)\times(n-r)}{\boldsymbol{F}} \end{bmatrix},$$

where $rank(\underset{r\times r}{\boldsymbol{C}}) = r$, then

$$\underset{n\times m}{\boldsymbol{G}} = \begin{bmatrix} \boldsymbol{C}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

is a GI of $\underset{m\times n}{\boldsymbol{A}}$.

Proof of Result A.10:

$$\boldsymbol{AG} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{F} \end{bmatrix} \begin{bmatrix} \boldsymbol{C}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{EC}^{-1} & \boldsymbol{0} \end{bmatrix}.$$

$$\therefore \boldsymbol{AGA} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{EC}^{-1} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{F} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{EC}^{-1}\boldsymbol{D} \end{bmatrix}.$$

We need to show $\boldsymbol{EC}^{-1}\boldsymbol{D} = \boldsymbol{F}$.

First note that

$$rank(\underset{r\times r}{\boldsymbol{C}}) = r \Longrightarrow rank([\boldsymbol{C},\boldsymbol{D}]) = r$$

$\because$ $[\boldsymbol{C},\boldsymbol{D}]$ has at least r LI columns and at most r LI rows.

$(rank([\boldsymbol{C},\boldsymbol{D}]) \geq r$ and $rank([\boldsymbol{C},\boldsymbol{D}]) \leq r \Longrightarrow rank([\boldsymbol{C},\boldsymbol{D}]) = r.)$

Now $rank\left(\begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{F} \end{bmatrix}\right) = r \Longrightarrow$ each row of $[\boldsymbol{E}, \boldsymbol{F}]$ is a LC of the rows of $[\boldsymbol{C}, \boldsymbol{D}]$.

Thus $\exists$ a matrix $\boldsymbol{K} \ni$

$$\begin{aligned} &\boldsymbol{K}[\boldsymbol{C}, \boldsymbol{D}] = [\boldsymbol{E}, \boldsymbol{F}] \\ \Longleftrightarrow\ &[\boldsymbol{KC}, \boldsymbol{KD}] = [\boldsymbol{E}, \boldsymbol{F}] \\ \Longleftrightarrow\ &\boldsymbol{KC} = \boldsymbol{E}, \boldsymbol{KD} = \boldsymbol{F}. \end{aligned}$$

Now $\boldsymbol{KC} = \boldsymbol{E} \iff \boldsymbol{K} = \boldsymbol{EC}^{-1}$. Together with $\boldsymbol{KD} = \boldsymbol{F}$, this implies $\boldsymbol{EC}^{-1}\boldsymbol{D} = \boldsymbol{F}$. $\square$

Permutation Matrix

A matrix $\underset{n\times n}{\boldsymbol{P}}$ is a permutation matrix if the rows of $\boldsymbol{P}$ are the same as the rows of $\underset{n\times n}{\boldsymbol{I}}$ but not necessarily in the same order.

Example:

- $\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

- If $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$, then $\boldsymbol{PA} = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 9 & 10 & 11 & 12 \end{bmatrix}$.

- Order of rows of $\boldsymbol{PA}$ are permuted relative to order of rows of $\boldsymbol{A}$.

Example:

- $\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$
- Then $\boldsymbol{BP} = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \end{bmatrix}.$
- Order of columns of $\boldsymbol{BP}$ are permuted relative to order of columns of $\boldsymbol{B}$.

A Permutation Matrix is Nonsingular

The rows (and columns) of a permutation matrix are the same as those of the identity matrix. Thus, a permutation matrix has full rank and is therefore nonsingular.

Furthermore, if $\boldsymbol{P} = [\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n]$ is a permutation matrix,

$$\boldsymbol{P}^{-1} = \boldsymbol{P}' \because \boldsymbol{p}_i'\boldsymbol{p}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Result A.11:

Suppose $\underset{m\times n}{rank(\boldsymbol{A})} = r$. There exist permutation matrices $\underset{m\times m}{\boldsymbol{P}}$ and $\underset{n\times n}{\boldsymbol{Q}}$ such that

$$\boldsymbol{PAQ} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{F} \end{bmatrix},$$

where $rank(\underset{r\times r}{\boldsymbol{C}}) = r$. Furthermore,

$$\boldsymbol{G} = \boldsymbol{Q} \begin{bmatrix} \boldsymbol{C}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \boldsymbol{P}$$

is a GI of $\boldsymbol{A}$.

Proof of Result A.11:

- Because $rank(\boldsymbol{A}) = r$, there exists a set of r rows of $\boldsymbol{A}$ that are LI.
- Let $\boldsymbol{P}$ be a permutation matrix, $\ni$ the first r rows of $\boldsymbol{PA}$ are LI.
- Let $\boldsymbol{H}$ be the matrix consisting of the first r rows of $\boldsymbol{PA}$. Then $rank(\boldsymbol{H}) = r$.

- This implies that $\exists$ a set of r columns of $\boldsymbol{H}$ that are LI.
- Let $\boldsymbol{Q}$ be a permutation matrix $\ni$ the first r columns of $\boldsymbol{HQ}$ are LI.
- Then the submatrix consisting of the first r rows and first r columns of $\boldsymbol{PAQ}$ has rank r.

- Thus we can partition $\boldsymbol{PAQ}$ as $\begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{F} \end{bmatrix}$, where $rank(\underset{r \times r}{\boldsymbol{C}}) = r$.

- By Result A.10, $\begin{bmatrix} \boldsymbol{C}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ is a GI for $\boldsymbol{PAQ} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{F} \end{bmatrix}$.

$$PAQ = \begin{bmatrix} C & D \\ E & F \end{bmatrix} \iff A = P^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} Q^{-1}$$

$$\therefore AQ \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} PA =$$

$$= P^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} Q^{-1} Q \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} PP^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} Q^{-1}$$

$$= P^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C & D \\ E & F \end{bmatrix} Q^{-1}$$

$$= P^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} Q^{-1} = A.$$

□

Use Result A.11 to find a GI for

$$A = \begin{bmatrix} 1 & 1 & 3 & 4 \\ 2 & 2 & 6 & 8 \\ 3 & 3 & 0 & 12 \end{bmatrix}.$$

- Let $\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Then $\boldsymbol{PA} = \begin{bmatrix} 1 & 1 & 3 & 4 \\ 3 & 3 & 0 & 12 \\ 2 & 2 & 6 & 8 \end{bmatrix}$.

- Let $\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

- Then $\boldsymbol{PAQ} = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 3 & 0 & 3 & 12 \\ 2 & 6 & 2 & 8 \end{bmatrix}$.

- Note that $\begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{9} \end{bmatrix}$.

- Thus, $\begin{bmatrix} 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{9} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is GI for $\boldsymbol{PAQ}$.

- GI for $\boldsymbol{A}$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{9} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{9} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \\ \frac{1}{3} & 0 & -\frac{1}{9} \\ 0 & 0 & 0 \end{bmatrix}.$$

To verify, note that

$$\boldsymbol{AG} = \underset{(\boldsymbol{A})}{\begin{bmatrix} 1 & 1 & 3 & 4 \\ 2 & 2 & 6 & 8 \\ 3 & 3 & 0 & 12 \end{bmatrix}} \underset{(\boldsymbol{G})}{\begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \\ \frac{1}{3} & 0 & -\frac{1}{9} \\ 0 & 0 & 0 \end{bmatrix}} = \underset{(\boldsymbol{AG})}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

and

$$(\boldsymbol{AG})\boldsymbol{A} = \underset{(\boldsymbol{AG})}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{(\boldsymbol{A})}{\begin{bmatrix} 1 & 1 & 3 & 4 \\ 2 & 2 & 6 & 8 \\ 3 & 3 & 0 & 12 \end{bmatrix}} = \underset{(\boldsymbol{AGA})}{\begin{bmatrix} 1 & 1 & 3 & 4 \\ 2 & 2 & 6 & 8 \\ 3 & 3 & 0 & 12 \end{bmatrix}}.$$

Algorithm for finding a GI of $\boldsymbol{A}$:

1. Find an $r \times r$ nonsingular submatrix of $\boldsymbol{A}$, where $r = rank(\boldsymbol{A})$. Call this matrix $\boldsymbol{W}$.

2. Compute $(\boldsymbol{W}^{-1})'$.

3. Replace each element of $\boldsymbol{W}$ in $\boldsymbol{A}$ with the corresponding elements of $(\boldsymbol{W}^{-1})'$.

4. Replace all other elements in $\boldsymbol{A}$ with zeros.

5. Transpose resulting matrix to get $\boldsymbol{A}^{-}$.

Result A.12:

Let $\boldsymbol{Ax} = \boldsymbol{c}$ be a consistent system of equations, and let $\boldsymbol{G}$ be any GI of $\boldsymbol{A}$. Then $\boldsymbol{Gc}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{c}$, i.e., $\boldsymbol{AGc} = \boldsymbol{c}$.

Proof of Result A.12:

- $\because \boldsymbol{Ax} = \boldsymbol{c}$ is consistent, $\exists$ a solution $\boldsymbol{x}^* \ni \boldsymbol{Ax}^* = \boldsymbol{c}$.
- $\therefore \boldsymbol{AGc} = \boldsymbol{AGAx}^* = \boldsymbol{Ax}^* = \boldsymbol{c}$. $\square$

Result A.13:

Let $\boldsymbol{Ax} = \boldsymbol{c}$ be a consistent system of equations, and let $\boldsymbol{G}$ be any GI of $\boldsymbol{A}$. Then $\tilde{\boldsymbol{x}}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{c}$ iff $\exists\, \boldsymbol{z} \ni \tilde{\boldsymbol{x}} = \boldsymbol{Gc} + (\boldsymbol{I} - \boldsymbol{GA})\boldsymbol{z}$.

Proof of Result A.13

$(\Longleftarrow)$:

$$\begin{aligned} & \boldsymbol{A}[\boldsymbol{G}\boldsymbol{c} + (\boldsymbol{I} - \boldsymbol{G}\boldsymbol{A})\boldsymbol{z}] \\ & = \boldsymbol{A}\boldsymbol{G}\boldsymbol{c} + (\boldsymbol{A} - \boldsymbol{A}\boldsymbol{G}\boldsymbol{A})\boldsymbol{z} \\ & = \boldsymbol{c} + (\boldsymbol{A} - \boldsymbol{A})\boldsymbol{z} \quad (\text{ by Result A.12}) \\ & = \boldsymbol{c}. \end{aligned}$$

$\therefore \boldsymbol{G}\boldsymbol{c} + (\boldsymbol{I} - \boldsymbol{G}\boldsymbol{A})\boldsymbol{z}$ is a solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}\ \forall\ \boldsymbol{z}$ (of appropriate dimension).

($\Longrightarrow$):

If $\boldsymbol{A}\tilde{\boldsymbol{x}} = \boldsymbol{c}$, then we have

$$\begin{aligned}
\tilde{\boldsymbol{x}} &= \boldsymbol{Gc} + \tilde{\boldsymbol{x}} - \boldsymbol{Gc} \\
&= \boldsymbol{Gc} + \tilde{\boldsymbol{x}} - \boldsymbol{GA}\tilde{\boldsymbol{x}} \\
&= \boldsymbol{Gc} + (\boldsymbol{I} - \boldsymbol{GA})\tilde{\boldsymbol{x}} \\
&\therefore \exists\, \boldsymbol{z}(\text{ namely } \boldsymbol{z} = \tilde{\boldsymbol{x}}) \ni \tilde{\boldsymbol{x}} = \boldsymbol{Gc} + (\boldsymbol{I} - \boldsymbol{GA})\boldsymbol{z}.
\end{aligned}$$

$\square$

Prove that a consistent system of equations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}$$

has a unique solution if and only if $\boldsymbol{A}$ is full-column rank and infinitely many solutions if and only if $\boldsymbol{A}$ is less than full-column rank.

Proof:

First, suppose $\boldsymbol{A}$ is full-column rank. Then,

$$\boldsymbol{A}\boldsymbol{w} = \boldsymbol{0} \iff \boldsymbol{w} = \boldsymbol{0}.$$

Suppose $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are any two solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}$. Then

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x}_1 = \boldsymbol{A}\boldsymbol{x}_2 &\iff \boldsymbol{A}\boldsymbol{x}_1 - \boldsymbol{A}\boldsymbol{x}_2 = \boldsymbol{0} \\ &\iff \boldsymbol{A}(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{0} \\ &\iff \boldsymbol{x}_1 - \boldsymbol{x}_2 = \boldsymbol{0} \\ &\iff \boldsymbol{x}_1 = \boldsymbol{x}_2. \end{aligned}$$

Thus, the solution to $\boldsymbol{Ax} = \boldsymbol{c}$ is unique when $\boldsymbol{A}$ has full-column rank.

Now suppose $rank(\underset{m \times n}{\boldsymbol{A}}) < n$.

Then $\exists\ \boldsymbol{w} \neq \boldsymbol{0} \ni \boldsymbol{Aw} = \boldsymbol{0}$.

Let $\boldsymbol{x}^*$ denote any solution to

$$\boldsymbol{A}\boldsymbol{x}^* = \boldsymbol{c}.$$

Then for any $d \in \mathbb{R}$,

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{x}^* + d\boldsymbol{w}) &= \boldsymbol{A}\boldsymbol{x}^* + \boldsymbol{A}d\boldsymbol{w} \\ &= \boldsymbol{A}\boldsymbol{x}^* + d\boldsymbol{A}\boldsymbol{w} \\ &= \boldsymbol{c} + d\boldsymbol{0} \\ &= \boldsymbol{c}. \end{aligned}$$

Thus, $\{\boldsymbol{x}^* + d\boldsymbol{w} : d \in \mathbb{R}\}$ is an infinite set of solutions for $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}$. $\square$