Introduction to the Gauss-Markov Linear Model

Random Vectors

- $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ is a random vector if and only if each element of $\boldsymbol{y}$ is a random variable (i.e., y_i is a random variable $\forall\ i = 1, \ldots, n$).

- The mean of the random vector $\boldsymbol{y}$ is $\mathrm{E}(\boldsymbol{y}) = \begin{bmatrix} \mathrm{E}(y_1) \\ \mathrm{E}(y_2) \\ \vdots \\ \mathrm{E}(y_n) \end{bmatrix}$.

- The variance of the random vector $\boldsymbol{y}$ is the matrix whose i, jth element is $\mathrm{Cov}(y_i, y_j) = \mathrm{E}(y_i y_j) - \mathrm{E}(y_i)\mathrm{E}(y_j)$.

Example: Variance of a Random Vector

For example, the variance of $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ is

$$\begin{aligned}
\text{Var}(\boldsymbol{y}) &= \begin{bmatrix} \text{Cov}(y_1, y_1) & \text{Cov}(y_1, y_2) & \text{Cov}(y_1, y_3) \\ \text{Cov}(y_2, y_1) & \text{Cov}(y_2, y_2) & \text{Cov}(y_2, y_3) \\ \text{Cov}(y_3, y_1) & \text{Cov}(y_3, y_2) & \text{Cov}(y_3, y_3) \end{bmatrix} \\
&= \begin{bmatrix} \text{Var}(y_1) & \text{Cov}(y_1, y_2) & \text{Cov}(y_1, y_3) \\ \text{Cov}(y_2, y_1) & \text{Var}(y_2) & \text{Cov}(y_2, y_3) \\ \text{Cov}(y_3, y_1) & \text{Cov}(y_3, y_2) & \text{Var}(y_3) \end{bmatrix}.
\end{aligned}$$

The Gauss-Markov Linear Model

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

- $\boldsymbol{y}$ is an $n \times 1$ random vector of responses.

- $\boldsymbol{X}$ is an $n \times p$ matrix of constants with columns corresponding to explanatory variables. $\boldsymbol{X}$ is sometimes referred to as the *design matrix*.

- $\boldsymbol{\beta}$ is an unknown parameter vector in $\mathbb{R}^p$.

- $\boldsymbol{\epsilon}$ is an $n \times 1$ random vector of errors.

- $\mathrm{E}(\boldsymbol{\epsilon}) = \mathbf{0}$ and $\mathrm{Var}(\boldsymbol{\epsilon}) = \sigma^2\boldsymbol{I}$, where σ^2 is an unknown parameter in $\mathbb{R}^+$.

The Gauss-Markov Linear Model

- Note that the model is not completely specified because the distribution of $\boldsymbol{y}$ is not completely specified.

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}, \qquad \mathrm{E}(\boldsymbol{\epsilon}) = \boldsymbol{0}, \qquad \mathrm{Var}(\boldsymbol{\epsilon}) = \sigma^2\boldsymbol{I}$$

$$\begin{aligned} &\Longrightarrow \quad \mathrm{E}(\boldsymbol{y}) = \boldsymbol{X\beta}, \qquad \mathrm{Var}(\boldsymbol{y}) = \sigma^2\boldsymbol{I} \\ &\Longrightarrow \quad \boldsymbol{y} \sim (\boldsymbol{X\beta}, \sigma^2\boldsymbol{I}) \end{aligned}$$

"$\boldsymbol{y}$ has a distribution with mean $\boldsymbol{X\beta}$ and variance $\sigma^2\boldsymbol{I}$."

The Normal Theory Gauss-Markov Linear Model

- We often add an assumption of multivariate normality to the Gauss-Markov linear model: $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$.
- The assumption $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$ is equivalent to $\epsilon_1, \ldots, \epsilon_n \overset{i.i.d.}{\sim} N(0, \sigma^2)$.
- The assumption $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I}) \implies \boldsymbol{y} \sim N(\boldsymbol{X\beta}, \sigma^2 \boldsymbol{I})$, i.e.,
  - $y_1, \ldots, y_n$ are independent normal random variables,
  - $\mathrm{Var(y_i)} = \sigma^2 \; \forall \; \mathrm{i} = 1, \ldots, \mathrm{n}$, and
  - $\mathrm{E(y_i)} = \boldsymbol{x}'_{(\mathrm{i})}\boldsymbol{\beta}$ (where $\boldsymbol{x}'_{(i)}$ is the ith row of $\boldsymbol{X}$) $\forall \; i = 1, \ldots, n$.

Goal of Analysis

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

- The goal of analysis often focuses on answering questions about certain linear functions of $\boldsymbol{\beta}$ of the form $\boldsymbol{C}\boldsymbol{\beta}$ for a specified matrix $\boldsymbol{C}$.
- The normality assumption is useful for constructing confidence intervals and performing tests concerning $\boldsymbol{C}\boldsymbol{\beta}$.

Example 1

Researchers harvested five randomly selected ears of corn from a field. For $i = 1, \ldots, 5$; let y_i denote the weight in grams of the i^{th} ear.

$$y_1, \ldots, y_5 \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$$

$$y_i = \mu + \epsilon_i, \quad i = 1, \ldots, 5; \quad \epsilon_1, \ldots, \epsilon_5 \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

$$\begin{aligned} y_1 &= \mu + \epsilon_1 \\ y_2 &= \mu + \epsilon_2 \\ y_3 &= \mu + \epsilon_3 \qquad\qquad \epsilon_1, \ldots, \epsilon_5 \overset{i.i.d.}{\sim} N(0, \sigma^2) \\ y_4 &= \mu + \epsilon_4 \\ y_5 &= \mu + \epsilon_5 \end{aligned}$$

Example 1 (continued)

$$
\begin{aligned}
y_1 &= \mu + \epsilon_1 \\
y_2 &= \mu + \epsilon_2 \\
y_3 &= \mu + \epsilon_3 \\
y_4 &= \mu + \epsilon_4 \\
y_5 &= \mu + \epsilon_5
\end{aligned}
\qquad \epsilon_1, \ldots, \epsilon_5 \overset{i.i.d.}{\sim} N(0, \sigma^2)
$$

$$
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} \mu \\ \mu \\ \mu \\ \mu \\ \mu \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}, \qquad \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})
$$

Example 1 (continued)

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} \mu \\ \mu \\ \mu \\ \mu \\ \mu \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}, \qquad \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [\mu] + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}, \qquad \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$$

Example 1 (continued)

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [\mu] + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}, \quad \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$$

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$$

$$\boldsymbol{C}\boldsymbol{\beta} = [1][\mu] = \mu$$

Example 2

Researchers randomly assigned eight experimental units to two treatments and measured a response of interest. For $i = 1, 2$; let $y_{i1}, y_{i2}, y_{i3}, y_{i4}$ denote the responses of the experimental units in the i^{th} treatment group.

$$y_{11}, y_{12}, y_{13}, y_{14} \overset{i.i.d.}{\sim} N(\mu_1, \sigma^2)$$

independent of

$$y_{21}, y_{22}, y_{23}, y_{24} \overset{i.i.d.}{\sim} N(\mu_2, \sigma^2)$$

$$y_{ij} = \mu_i + \epsilon_{ij}, \quad i = 1, 2;\ j = 1, \ldots, 4$$

$$\epsilon_{11}, \epsilon_{12}, \epsilon_{13}, \epsilon_{14}, \epsilon_{21}, \epsilon_{22}, \epsilon_{23}, \epsilon_{24} \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

Example 2 (continued)

$$\begin{aligned}
y_{11} &= \mu_1 + \epsilon_{11} \\
y_{12} &= \mu_1 + \epsilon_{12} \\
y_{13} &= \mu_1 + \epsilon_{13} \\
y_{14} &= \mu_1 + \epsilon_{14} \\
y_{21} &= \mu_2 + \epsilon_{21} \\
y_{22} &= \mu_2 + \epsilon_{22} \\
y_{23} &= \mu_2 + \epsilon_{23} \\
y_{24} &= \mu_2 + \epsilon_{24}
\end{aligned}$$

$$\epsilon_{11}, \epsilon_{12}, \epsilon_{13}, \epsilon_{14}, \epsilon_{21}, \epsilon_{22}, \epsilon_{23}, \epsilon_{24} \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

Example 2 (continued)

$$\begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{21} \\ y_{22} \\ y_{23} \\ y_{24} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_1 \\ \mu_1 \\ \mu_1 \\ \mu_2 \\ \mu_2 \\ \mu_2 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{14} \\ \epsilon_{21} \\ \epsilon_{22} \\ \epsilon_{23} \\ \epsilon_{24} \end{bmatrix}, \qquad \begin{bmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{14} \\ \epsilon_{21} \\ \epsilon_{22} \\ \epsilon_{23} \\ \epsilon_{24} \end{bmatrix} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$$

Example 2 (continued)

$$\begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{21} \\ y_{22} \\ y_{23} \\ y_{24} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{14} \\ \epsilon_{21} \\ \epsilon_{22} \\ \epsilon_{23} \\ \epsilon_{24} \end{bmatrix}, \qquad \begin{bmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{14} \\ \epsilon_{21} \\ \epsilon_{22} \\ \epsilon_{23} \\ \epsilon_{24} \end{bmatrix} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$$

Example 2 (continued)

$$\begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{21} \\ y_{22} \\ y_{23} \\ y_{24} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{14} \\ \epsilon_{21} \\ \epsilon_{22} \\ \epsilon_{23} \\ \epsilon_{24} \end{bmatrix}, \quad \begin{bmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{14} \\ \epsilon_{21} \\ \epsilon_{22} \\ \epsilon_{23} \\ \epsilon_{24} \end{bmatrix} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$$

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$$

$$\boldsymbol{C}\boldsymbol{\beta} = [1, -1] \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \mu_1 - \mu_2$$

Example 3

Suppose eight fertilizer amounts denoted $x_1, \ldots, x_8$ were randomly assigned to eight field plots. For $i = 1, \ldots, 8$; let y_i denote the yield of the plot that received fertilizer amount x_i.

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad i = 1, \ldots, 8$$

$$\epsilon_1, \ldots, \epsilon_8 \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

$$\begin{aligned}
y_1 &= \beta_0 + \beta_1 x_1 + \epsilon_1 \\
y_2 &= \beta_0 + \beta_1 x_2 + \epsilon_2 \\
y_3 &= \beta_0 + \beta_1 x_3 + \epsilon_3 \\
y_4 &= \beta_0 + \beta_1 x_4 + \epsilon_4 \\
y_5 &= \beta_0 + \beta_1 x_5 + \epsilon_5 \\
y_6 &= \beta_0 + \beta_1 x_6 + \epsilon_6 \\
y_7 &= \beta_0 + \beta_1 x_7 + \epsilon_7 \\
y_8 &= \beta_0 + \beta_1 x_8 + \epsilon_8
\end{aligned}$$

Example 3 (continued)

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} \beta_0 + \beta_1 x_1 \\ \beta_0 + \beta_1 x_2 \\ \beta_0 + \beta_1 x_3 \\ \beta_0 + \beta_1 x_4 \\ \beta_0 + \beta_1 x_5 \\ \beta_0 + \beta_1 x_6 \\ \beta_0 + \beta_1 x_7 \\ \beta_0 + \beta_1 x_8 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \end{bmatrix}, \qquad \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \end{bmatrix} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$$

Example 3 (continued)

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \\ 1 & x_5 \\ 1 & x_6 \\ 1 & x_7 \\ 1 & x_8 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \end{bmatrix}, \qquad \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \end{bmatrix} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$$

Example 3 (continued)

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \\ 1 & x_5 \\ 1 & x_6 \\ 1 & x_7 \\ 1 & x_8 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \end{bmatrix}, \qquad \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \end{bmatrix} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$$

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$$

$$\boldsymbol{C\beta} = [0, 1] \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \beta_1$$

Example 4

Eight hogs were randomly assigned to two diets and two inoculations such that two hogs received each combination of diet and inoculation.

- This experiment involves two *factors*: **diet** and **inoculation**.
- In this case, each factor has two *levels* (denoted here generically as 1 and 2).
- A combination of one level from each factor forms a *treatment*.
- In this case, we have four treatments:

Treatment	Diet	Inoculation
1	1	1
2	1	2
3	2	1
4	2	2

Example 4 (continued)

For $i = 1, 2$; $j = 1, 2$; and $k = 1, 2$; let y_{ijk} denote the average daily gain of the k^{th} hog that received diet i and inoculation j.

$$y_{ijk} = \mu + \epsilon_{ijk} \quad i = 1, 2; j = 1, 2; k = 1, 2;$$

$$\epsilon_{111}, \epsilon_{112}, \epsilon_{121}, \epsilon_{122}, \epsilon_{211}, \epsilon_{212}, \epsilon_{221}, \epsilon_{222} \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

Under this model, neither diet nor inoculation affects average daily gain.

Example 4 (continued)

For $i = 1, 2$; $j = 1, 2$; and $k = 1, 2$; let y_{ijk} denote the average daily gain of the k^{th} hog that received diet i and inoculation j.

$$y_{ijk} = \mu + \alpha_i + \epsilon_{ijk} \quad i = 1, 2; j = 1, 2; k = 1, 2;$$

$$\epsilon_{111}, \epsilon_{112}, \epsilon_{121}, \epsilon_{122}, \epsilon_{211}, \epsilon_{212}, \epsilon_{221}, \epsilon_{222} \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

Under this model, only diet affects average daily gain.

Example 4 (continued)

For $i = 1, 2$; $j = 1, 2$; and $k = 1, 2$; let y_{ijk} denote the average daily gain of the k^{th} hog that received diet i and inoculation j.

$$y_{ijk} = \mu + \beta_j + \epsilon_{ijk} \quad i = 1, 2; j = 1, 2; k = 1, 2;$$

$$\epsilon_{111}, \epsilon_{112}, \epsilon_{121}, \epsilon_{122}, \epsilon_{211}, \epsilon_{212}, \epsilon_{221}, \epsilon_{222} \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

Under this model, only inoculation affects average daily gain.

Example 4 (continued)

$$y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk} \quad i = 1, 2; j = 1, 2; k = 1, 2;$$

$$\epsilon_{111}, \epsilon_{112}, \epsilon_{121}, \epsilon_{122}, \epsilon_{211}, \epsilon_{212}, \epsilon_{221}, \epsilon_{222} \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

- Under this model, *factors* **diet** and **inoculation** affect the mean average daily gain in an *additive* manner.
- There is no *interaction* between the *factors* **diet** and **inoculation**.

	inoculation		
diet	1	2	inoculation difference
1	$\mu + \alpha_1 + \beta_1$	$\mu + \alpha_1 + \beta_2$	$\beta_1 - \beta_2$
2	$\mu + \alpha_2 + \beta_1$	$\mu + \alpha_2 + \beta_2$	$\beta_1 - \beta_2$
diet difference	$\alpha_1 - \alpha_2$	$\alpha_1 - \alpha_2$	

Example 4 (continued)

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk} \quad i = 1, 2; j = 1, 2; k = 1, 2;$$

$$\epsilon_{111}, \epsilon_{112}, \epsilon_{121}, \epsilon_{122}, \epsilon_{211}, \epsilon_{212}, \epsilon_{221}, \epsilon_{222} \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

- Under this model, there is one mean for each combination of diet and inoculation.
- Those four means are free to take any four values with no restrictions.

	inoculation		
diet	1	2	Δinoculation
1	$\mu + \alpha_1 + \beta_1 + \gamma_{11}$	$\mu + \alpha_1 + \beta_2 + \gamma_{12}$	$\beta_1 - \beta_2 + \gamma_{11} - \gamma_{12}$
2	$\mu + \alpha_2 + \beta_1 + \gamma_{21}$	$\mu + \alpha_2 + \beta_2 + \gamma_{22}$	$\beta_1 - \beta_2 + \gamma_{21} - \gamma_{22}$
Δdiet	$\alpha_1 - \alpha_2 + \gamma_{11} - \gamma_{21}$	$\alpha_1 - \alpha_2 + \gamma_{12} - \gamma_{22}$	

Example 4 (continued)

An equivalent model is the so called *cell means* model:

$$y_{ijk} = \mu_{ij} + \epsilon_{ijk} \quad i = 1, 2; j = 1, 2; k = 1, 2;$$

$$\epsilon_{111}, \epsilon_{112}, \epsilon_{121}, \epsilon_{122}, \epsilon_{211}, \epsilon_{212}, \epsilon_{221}, \epsilon_{222} \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

	inoculation		
diet	1	2	Δinoculation
1	μ_{11}	μ_{12}	$\mu_{11} - \mu_{12}$
2	μ_{21}	μ_{22}	$\mu_{21} - \mu_{22}$
Δdiet	$\mu_{11} - \mu_{21}$	$\mu_{12} - \mu_{22}$	

Example 4 (continued)

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk} \quad i = 1, 2; j = 1, 2; k = 1, 2;$$

$$\begin{aligned}
y_{111} &= \mu + \alpha_1 + \beta_1 + \gamma_{11} + \epsilon_{111} \\
y_{112} &= \mu + \alpha_1 + \beta_1 + \gamma_{11} + \epsilon_{112} \\
y_{121} &= \mu + \alpha_1 + \beta_2 + \gamma_{12} + \epsilon_{121} \\
y_{122} &= \mu + \alpha_1 + \beta_2 + \gamma_{12} + \epsilon_{122} \\
y_{211} &= \mu + \alpha_2 + \beta_1 + \gamma_{21} + \epsilon_{211} \\
y_{212} &= \mu + \alpha_2 + \beta_1 + \gamma_{21} + \epsilon_{212} \\
y_{221} &= \mu + \alpha_2 + \beta_2 + \gamma_{22} + \epsilon_{221} \\
y_{222} &= \mu + \alpha_2 + \beta_2 + \gamma_{22} + \epsilon_{222}
\end{aligned}$$

$$\epsilon_{111}, \epsilon_{112}, \epsilon_{121}, \epsilon_{122}, \epsilon_{211}, \epsilon_{212}, \epsilon_{221}, \epsilon_{222} \overset{i.i.d.}{\sim} N(0, \sigma^2)$$

Example 4 (continued)

$$\begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \end{bmatrix} = \begin{bmatrix} \mu + \alpha_1 + \beta_1 + \gamma_{11} \\ \mu + \alpha_1 + \beta_1 + \gamma_{11} \\ \mu + \alpha_1 + \beta_2 + \gamma_{12} \\ \mu + \alpha_1 + \beta_2 + \gamma_{12} \\ \mu + \alpha_2 + \beta_1 + \gamma_{21} \\ \mu + \alpha_2 + \beta_1 + \gamma_{21} \\ \mu + \alpha_2 + \beta_2 + \gamma_{22} \\ \mu + \alpha_2 + \beta_2 + \gamma_{22} \end{bmatrix} + \begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \end{bmatrix}, \begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \end{bmatrix} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$$

Example 4 (continued)

$$\begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{21} \\ \gamma_{22} \end{bmatrix} + \begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \end{bmatrix}$$

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim N(\boldsymbol{0}, \sigma^2\boldsymbol{I})$$

Example 4 (continued)

$$\boldsymbol{\beta} = [\mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}]'$$

	inoculation	
diet	1	2
1	$\mu + \alpha_1 + \beta_1 + \gamma_{11}$	$\mu + \alpha_1 + \beta_2 + \gamma_{12}$
2	$\mu + \alpha_2 + \beta_1 + \gamma_{21}$	$\mu + \alpha_2 + \beta_2 + \gamma_{22}$
Δdiet	$\alpha_1 - \alpha_2 + \gamma_{11} - \gamma_{21}$	$\alpha_1 - \alpha_2 + \gamma_{12} - \gamma_{22}$

Is the difference between diet means for inoculation 1 the same as the difference between diet means for inoculation 2?

$$\boldsymbol{C\beta} = [0, 0, 0, 0, 0, 1, -1, -1, 1]\boldsymbol{\beta} = \gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22} = 0?$$

This questions asks if there is *interaction* between the *factors* **diet** and **inoculation.**

Example 4 (continued)

$$\boldsymbol{\beta} = [\mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}]'$$

	inoculation		
diet	1	2	Δinoculation
1	$\mu + \alpha_1 + \beta_1 + \gamma_{11}$	$\mu + \alpha_1 + \beta_2 + \gamma_{12}$	$\beta_1 - \beta_2 + \gamma_{11} - \gamma_{12}$
2	$\mu + \alpha_2 + \beta_1 + \gamma_{21}$	$\mu + \alpha_2 + \beta_2 + \gamma_{22}$	$\beta_1 - \beta_2 + \gamma_{21} - \gamma_{22}$

Is the difference between inoculation means for diet 1 the same as the difference between inoculation means for diet 2?

$$\boldsymbol{C\beta} = [0, 0, 0, 0, 0, 1, -1, -1, 1]\boldsymbol{\beta} = \gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22} = 0?$$

This questions also asks if there is *interaction* between the *factors* **diet** and **inoculation.**

Example 4 (continued)

$$\boldsymbol{\beta} = [\mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}]'$$

	inoculation		
diet	1	2	Diet Means
1	$\mu + \alpha_1 + \beta_1 + \gamma_{11}$	$\mu + \alpha_1 + \beta_2 + \gamma_{12}$	$\mu + \alpha_1 + \bar{\beta}_{\cdot} + \bar{\gamma}_{1\cdot}$
2	$\mu + \alpha_2 + \beta_1 + \gamma_{21}$	$\mu + \alpha_2 + \beta_2 + \gamma_{22}$	$\mu + \alpha_2 + \bar{\beta}_{\cdot} + \bar{\gamma}_{2\cdot}$

Is the average over inoculation means for diet 1 different than the average over inoculation means for diet 2?

$$\boldsymbol{C\beta} = [0, 1, -1, 0, 0, .5, .5, -.5, -.5]\boldsymbol{\beta} = \alpha_1 - \alpha_2 + \bar{\gamma}_{1\cdot} - \bar{\gamma}_{2\cdot} = 0?$$

This question asks about the *main effect* of the *factor* **diet.**

Example 4 (continued)

$$\boldsymbol{\beta} = [\mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}]'$$

	inoculation	
diet	1	2
1	$\mu + \alpha_1 + \beta_1 + \gamma_{11}$	$\mu + \alpha_1 + \beta_2 + \gamma_{12}$
2	$\mu + \alpha_2 + \beta_1 + \gamma_{21}$	$\mu + \alpha_2 + \beta_2 + \gamma_{22}$
Inoculation Means	$\mu + \bar{\alpha}. + \beta_1 + \bar{\gamma}_{\cdot 1}$	$\mu + \bar{\alpha}. + \beta_2 + \bar{\gamma}_{\cdot 2}$

Is the average over diet means for inoculation 1 different than the average over diet means for inoculation 2?

$$\boldsymbol{C\beta} = [0, 0, 0, 1, -1, .5, -.5, .5, -.5]\boldsymbol{\beta} = \beta_1 - \beta_2 + \bar{\gamma}_{\cdot 1} - \bar{\gamma}_{\cdot 2} = 0?$$

This question asks about the *main effect* of the *factor* **inoculation.**

Example 4 (continued)

$$\boldsymbol{\beta} = [\mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}]'$$

	inoculation	
diet	1	2
1	$\mu + \alpha_1 + \beta_1 + \gamma_{11}$	$\mu + \alpha_1 + \beta_2 + \gamma_{12}$
2	$\mu + \alpha_2 + \beta_1 + \gamma_{21}$	$\mu + \alpha_2 + \beta_2 + \gamma_{22}$
Δdiet	$\alpha_1 - \alpha_2 + \gamma_{11} - \gamma_{21}$	

Is there a difference between the diet means for inoculation 1?

$$\boldsymbol{C\beta} = [0, 1, -1, 0, 0, 1, 0, -1, 0]\boldsymbol{\beta} = \alpha_1 - \alpha_2 + \gamma_{11} - \gamma_{21} = 0?$$

This question asks about the *simple effect* of the *factor* **diet** for the first *level* of the *factor* **inoculation.**

Example 4 (continued)

$$\boldsymbol{\beta} = [\mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}]'$$

	inoculation		
diet	1	2	Δinoculation
1	$\mu + \alpha_1 + \beta_1 + \gamma_{11}$	$\mu + \alpha_1 + \beta_2 + \gamma_{12}$	$\beta_1 - \beta_2 + \gamma_{11} - \gamma_{12}$
2	$\mu + \alpha_2 + \beta_1 + \gamma_{21}$	$\mu + \alpha_2 + \beta_2 + \gamma_{22}$	$\beta_1 - \beta_2 + \gamma_{21} - \gamma_{22}$
Δdiet	$\alpha_1 - \alpha_2 + \gamma_{11} - \gamma_{21}$	$\alpha_1 - \alpha_2 + \gamma_{12} - \gamma_{22}$	

Are all four treatment means identical?

$$\begin{aligned} \boldsymbol{C\beta} &= \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \boldsymbol{\beta} \\ &= \begin{bmatrix} \beta_1 - \beta_2 + \gamma_{11} - \gamma_{12} \\ \beta_1 - \beta_2 + \gamma_{21} - \gamma_{22} \\ \alpha_1 - \alpha_2 + \gamma_{11} - \gamma_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ? \end{aligned}$$