Some Chapter 2 Notes on Completely Randomized Designs
(continued)

Recall that we can write the model for data from a completely randomized design as

$$y_{ij} = \mu_i + e_{ij} \quad i = 1, \ldots, t \quad j = 1, \ldots, r_i \quad e_{ij} \sim \text{ independent } N(0, \sigma^2)$$

where y_{ij} denotes the jth observation from the ith treatment group, μ_i denotes the mean of the ith treatment group, e_{ij} denotes the experimental error for the jth observation from the ith treatment group, t is the number of treatments, and r_i denotes the number of replications in the ith treatment group.

Treatment Effects Version of the Model

Suppose that we have a balanced design with r replications in each treatment group.

Let $\mu = \bar{\mu}. = \frac{1}{t}\sum_{i=1}^{t}\mu_i$

Let $\tau_i = \mu_i - \bar{\mu}.$ for $i = 1, \ldots, t$

Then $y_{ij} = \mu + \tau_i + e_{ij} \quad i = 1, \ldots, t \quad j = 1, \ldots, r \quad e_{ij} \sim$ independent $N(0, \sigma^2)$

τ_i is called the effect of treatment i or the ith treatment effect.

By the way we defined $\tau_1, \ldots, \tau_t$, the treatment effects add up to zero ($\sum_{i=1}^{t}\tau_i = 0$).

Our hypotheses become $H_0 : \tau_1 = \cdots = \tau_t = 0$ vs. $H_A : \tau_i \neq 0$ for some i

$\hat{\tau}_i = \hat{\mu}_i - \hat{\mu} = \bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}$ is the least-squares estimate of $\tau_i = \mu_i - \bar{\mu}_{\cdot\cdot}$.

$\text{SSTreatment} = \sum_{i=1}^{t} r(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 = \sum_{i=1}^{t} r\hat{\tau}_i^2 = r\sum_{i=1}^{t}\hat{\tau}_i^2$

Note that it makes sense to reject H_0 when SSTreatment is big.

Expected Mean Squares

$\text{Mean(MSTreatment)} = E(\text{MSTreatment}) = \sigma^2 + r\theta_t^2$ where $\theta_t^2 = \frac{1}{t-1}\sum_{i=1}^{t}\tau_i^2$

$\text{Mean(MSError)} = E(\text{MSError}) = \sigma^2$

Recall that our test statistic for testing $H_0 : \tau_1 = \cdots = \tau_t = 0$ vs. $H_A : \tau_i \neq 0$ for some i is $F = \frac{\text{MSTreatment}}{\text{MSError}}$.

Note that $\theta_t^2 = 0$ if H_0 is true and $\theta_t^2 > 0$ if H_0 is false.

From the expected mean squares, we can see that MSTreatment $\approx$ MSError when H_0 is true, and MSTreatment $>$ MSError when H_0 is false.

Thus $F = \frac{\text{MSTreatment}}{\text{MSError}} \approx 1$ when H_0 is false, and $F = \frac{\text{MSTreatment}}{\text{MSError}} > 1$.

When H_0 is true $F = \frac{\text{MSTreatment}}{\text{MSError}}$ takes random values around 1 according to an F distribution with $t-1$ and $N-t$ degrees of freedom.

We decide not to believe H_0 if the observed value of $F = \frac{\text{MSTreatment}}{\text{MSError}}$ is farther above 1 than we would expect an F random variable with $t-1$ and $N-t$ degrees of freedom to be.

For example, if the chance is 0.05 or less that an F random variable with $t-1$ and $N-t$ degrees of freedom would be as large or larger than the value of $F = \frac{\text{MSTreatment}}{\text{MSError}}$ we computed from the data, we often choose to doubt the null hypothesis and believe the alternative.

Power of the F-Test

When H_A is true, $F = \frac{\text{MSTreatment}}{\text{MSError}}$ takes random values according to a

$$\text{non-central } F\text{-distribution with } t-1 \text{ and } N-t \text{ d.f. and non-centrality parameter } \lambda = \frac{r\sum_{i=1}^{t}\tau_i^2}{\sigma^2}$$

Suppose there are three treatments with means $\mu_1 = 31$, $\mu_2 = 26$, and $\mu_3 = 33$. Suppose the variance of the normally distributed responses within each treatment group is 10. What is the distribution of the F-statistic for testing $H_0 : \mu_1 = \mu_2 = \mu_3$ if a completely randomized design is used with 4 replications for each treatment?

Type I Error=Reject H_0 when H_0 is true.

Type II Error=Accept H_0 when H_0 is false.

The type I error rate, denoted by α, is the probability of making a type I error when conducting a test with a true null hypothesis.

Suppose you are going to conduct an experiment and test a null hypothesis that (unbeknownst to you) is true. Suppose you will decide to reject the null hypothesis if the p-value that you will compute is less than 0.05. (Such a test is said to have significance level 0.05.) What is the probability that you will make a type I error, i.e., what is α?

β is the probability of making a type II error when conducting a test with a false null hypothesis.

The *power* of a test is the probability of rejecting H_0 when H_0 is false. The power of a test is equal to $1-\beta$.

What can a scientist do to increase the power of a test?

Use the SAS program *power.sas* to find the power of the F-test of $H_0 : \mu_1 = \mu_2 = \mu_3$ in the completely randomized design with $\mu_1 = 31$, $\mu_2 = 26$, $\mu_3 = 33$, $\sigma^2 = 10$ and 4 replications for each treatment.